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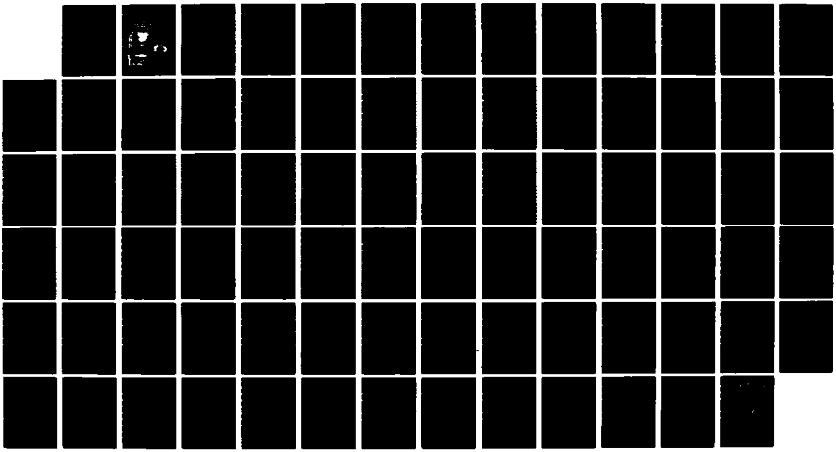

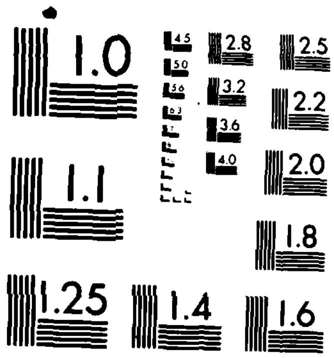

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MINIMAX DECISION PROBLEMS FOR GAUSSIAN TEST CHANNELS IN THE PRESENCE OF UNKNOWN JAMMING NOISE

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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MINIMAX DECISION PROBLEMS FOR GAUSSIAN TEST CHANNELS IN THE PRESENCE OF UNKNOWN JAMMING NOISE

BY

YING WAH WU

B.S., Columbia University, 1981

THESIS

Submitted in partial fulfillment of the requirements
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in the Graduate College of the
University of Illinois at Urbana-Champaign, 1984

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O 517

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TABLE OF CONTENTS

CHAPTER 1

PROBLEM FORMULATION

1.1. Introduction

In this thesis, we study solutions to a class of communication problems which involve transmission of a sequence of input variables over noisy channels under a given power constraint and in the presence of an intelligent jammer. In the next few sections, precise formulations of the problems will be presented. Generally, either the saddle-point solutions (Def. 1.1), or, (if it does not exist) the worst-case solutions by both min.max. and max.min. approaches will be obtained for each problem.

In Chapter 2, we consider a class of scalar problems, and determine the corresponding policies explicitly in analytic form (either saddle-point solutions or worst-case solutions). Then, we summarize all the results pertaining to the scalar problem in Table 2.1.

In Chapter 3, we treat vector versions of the problems considered in Chapter 2. First we obtain solutions for two classes of vector problems explicitly in analytic form, and when explicit solutions are not possible we develop a general recursive procedure for computing the corresponding solutions.

Chapter 4 discusses some extensions of the theory presented in this thesis, proposes topics for further research, and provides a general summary of the totality of the results obtained in the thesis. After that, all references, [1]-[8], are listed on the last page of this thesis.

1.2. Complete Description of the Scalar Problems

The general class of jamming problems to be treated in this thesis admit two scalar versions.

Version 1: The first scalar version is completely described by the communication system depicted in Fig. 1.1. The main communication channel is additive and memoryless, with an additive jamming noise, y, and a Gaussian noise, w. The noiseless side channel permits information to flow in the forward direction from the transmitter to the receiver under the restriction that only the information about the current structure of the encoding policy of the transmitter is allowed to be sent over it. The problem is to transmit the input signal, u, reliably through the main communication channel to the receiver, in the presence of the intelligent jammer (whose role is yet to be clarified) and the Gaussian noise, w, with the aid of the side channel. This will entail an optimum design of encoders and decoders as to be elucidated in the sequel. We now first provide a precise mathematical description of the system depicted in Fig. 1.1.

The input signal, u, is a sequence of independent, identically distributed Gaussian random variables with zero mean and unit variance. The transmitter converts the input signal, u, into the variable, x, with the encoding policy $\gamma(u)$ being an element of the space Γ_e of random mappings satisfying the power constraint $E\{(\gamma(u)^2\} \leq c^2$. By adopting a worst-case analysis, we assume that the jamming noise could be correlated with the message, u, and therefore let $y = \beta(u)$, where the jammer policy β is chosen from the space Γ_j of random mappings satisfying the power constraint, $E(\beta^2(u)) \leq k^2$. Finally, the channel noise, w, is a sequence of independent,

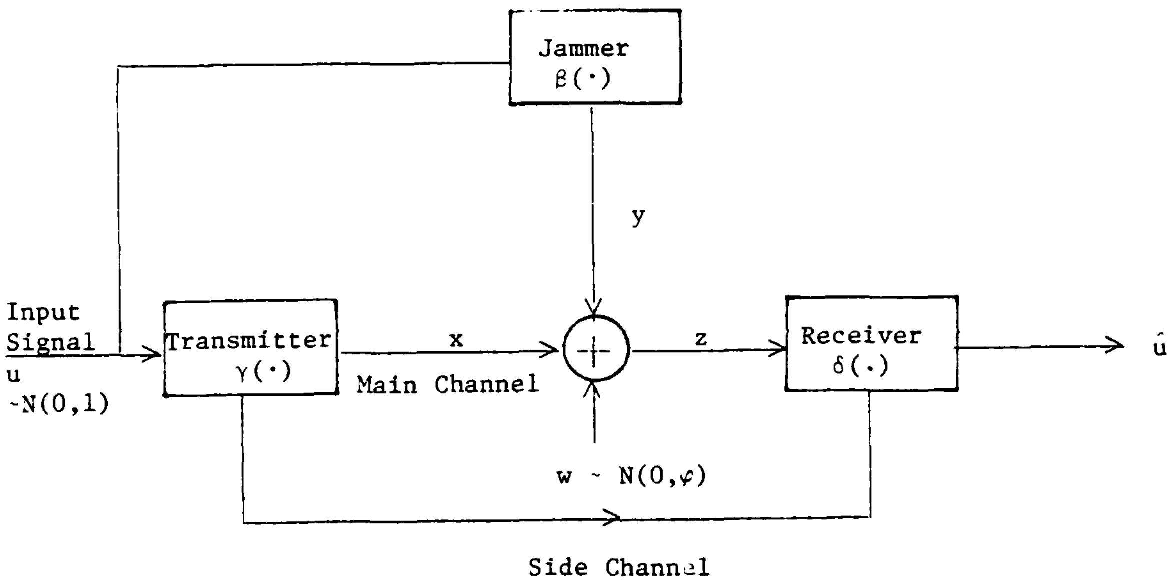

Figure 1.1 The Gaussian test channel for version (i).

identically distributed Gaussian random variables with zero mean and variance $\varphi > 0$, and w is independent of u, x, and y. Now the channel output to the receiver will be

$$z = x + y + w$$

The estimator of the receiver, $\delta(z)$ seeks to estimate the input signal based on the measurement of z under a mean square error criterion where $\delta(\cdot)$ is, in general, an element of the space Γ_α of random mappings from $\mathbb{R}$ into $\mathbb{R}$.

Version 2: The second scalar version differs from the previous version only in the sense that the side channel is now absent (depicted later in Chapter 2).

For future reference, we will call the first and second versions as case (i) and case (ii), respectively. The following mathematical formulation will apply to both cases.

Both the transmitter-receiver pair and the jammer seek to optimize the same criterion, $J(\gamma,\delta,\beta)$, which is the expected mean square error of estimating the input signal, u. Algebraically,

$$J(\gamma,\delta,\beta) = E\{(u - \delta(x + y + w))^2\}$$

The roles, however, are different; while the jammer wants to maximize J, the transmitter-receiver pair tries to minimize the same criterion. Because of this conflict of interest, we seek to obtain either a saddle-point solution, or, if it does not exist, the worst case solutions by min.max. and max.min. approaches.

In the min.max. approach, we proceed to evaluate the following expression

$$\bar{J}^* = \min_{\substack{\gamma\in\Gamma_e \\ \delta\in\Gamma_d}} \max_{\beta\in\Gamma_j} J(\gamma,\delta,\beta) \tag{1.1}$$

To evaluate $\bar{J}^*$, first the maximization of $J(\gamma,\delta,\beta)$ over the space Γ_j is carried out for each possible pair of encoding and decoding policies, (γ,δ), in the space of the Cartesian product of the two spaces, Γ_e and Γ_d. Hence, for each fixed pair, (γ,δ), we should obtain (assuming that such a solution exists)

$$\beta^*_{(\gamma,\delta)} = \arg. \max_{\beta\in\Gamma_j} J(\gamma,\delta,\beta) \tag{1.2}$$

equivalently,

$$J(\gamma,\delta,\beta^*_{(\gamma,\delta)}) \geq J(\gamma,\delta,\beta) \quad , \quad \forall\beta\in\Gamma_j$$

The next step is to pick out a pair (γ^*,δ^*) from the space $\Gamma_e \times \Gamma_d$ such that it minimizes $J(\gamma,\delta,\beta^*_{(\gamma,\delta)})$. Algebraically,

$$(\gamma^*,\delta^*) = \arg. \min_{(\gamma,\delta)\in\Gamma_e\times\Gamma_d} J(\gamma,\delta,\beta^*_{(\gamma,\delta)}) \tag{1.3}$$

Equivalently,

$$\bar{J}^* = J(\gamma^*,\delta^*,\beta^*_{(\gamma^*,\delta^*)}) \leq J(\gamma,\delta,\beta^*_{(\gamma,\delta)}) \ , \ \forall(\gamma,\delta)\in\Gamma_e\times\Gamma_d$$

This is the so-called *upper value* of the zero-sum game with kernel J.

For the max.min. approach we simply interchange the order of the maximization and minimization operations:

$$\underline{J}^* = \max_{\beta \in \Gamma_j} \; \min_{\substack{\gamma \in \Gamma_e \\ \delta \in \Gamma_d}} \; J(\gamma,\delta,\beta) \tag{1.4}$$

Firstly, we minimize $J(\gamma,\delta,\beta)$ with respect to (γ,δ) for each fixed jamming policy $\beta(\cdot)$. Let us define, for fixed $\beta(\cdot)$,

$$(\gamma^*_\beta,\delta^*_\beta) = \arg \min_{(\gamma,\delta) \in \Gamma_e \times \Gamma_d} J(\gamma,\delta,\beta) \tag{1.5}$$

equivalently,

$$J(\gamma^*_\beta,\delta^*_\beta,\beta) \le J(\gamma,\delta,\beta) \quad , \; \forall(\gamma,\delta) \in \Gamma_e \times \Gamma_d$$

Then, $J(\gamma^*_\beta,\delta^*_\beta,\beta)$ is maximized over the space Γ_j. So we define

$$\beta^* = \arg. \min_{\beta \in \Gamma_j} J(\gamma^*_\beta,\delta^*_\beta,\beta) \tag{1.6}$$

equivalently,

$$\underline{J}^* = J(\gamma^*_{\beta^*},\delta^*_{\beta^*},\beta^*) \ge J(\gamma^*_\beta,\delta^*_\beta,\beta) \quad , \; \forall\beta \in \Gamma_j$$

This is the so-called *lower value* of the zero-sum game whose kernel is J.

<u>Definition 1.1</u>: If there exists a triple $(\gamma^*,\delta^*,\beta^*) \in \Gamma_e \times \Gamma_d \times \Gamma_j$ satisfying the set of inequalities

$$J(\gamma^*,\delta^*,\beta) \le J(\gamma^*,\delta^*,\beta^*) \le J(\gamma,\delta,\beta^*) \quad , \; \forall\gamma \in \Gamma_e, \; \delta \in \Gamma_d, \; \beta \in \Gamma_j \tag{1.7}$$

then $(\gamma^*,\delta^*,\beta^*)$ is a saddle-point solution, and $J(\gamma^*,\delta^*,\beta^*)$ is the saddle-point value. □

<u>Lemma 1.1 [2]</u>: A saddle-point solution exists if and only if $\bar{J}^* = \underline{J}^* = J^*$, in which case $J(\gamma^*,\delta^*,\beta^*) = J^*$.

1.3. Review of Recent Literature

The recent result in the literature which is closely related to our work here is the one obtained by Başar [3]. The communication system treated in his paper is depicted below in Fig. 1.2. The three channel noises present in Fig. 1.2., w_1, w_2, v, are Gaussian and independent of each other. Further, w_1 and v are independent of x and u, and w_2 is independent of x, u, and v. The jammer feeds back a jamming noise to the channel based on a noisy version of the encoded signal, $y = x + v + w_1$. Other aspects of the communication system in [3] are the same as those we have introduced earlier in Section 1.2.

In order to be able to compare the results of [3] with the ones to be obtained in this thesis, we simplify the communication system of Fig. 1.2 by setting $\zeta_1 = \sigma = 0$, $\zeta_2 = \varphi$, and calling $w_2 = w$. With these simplifications, the variable, y, is no longer distinguishable from x, which we henceforth call x. Furthermore, to be consistent with the notation of Fig. 1.1, we replace ν of Fig. 1.2 with y. With this new notation, the communication system of Fig. 1.2 has been reproduced below as Fig. 1.3. Exact expressions for the saddle-point solution of the communication system of Fig. 1.3 will be summarized in Section 2.1 as Theorem 1. For future reference we will call this problem case (iii).

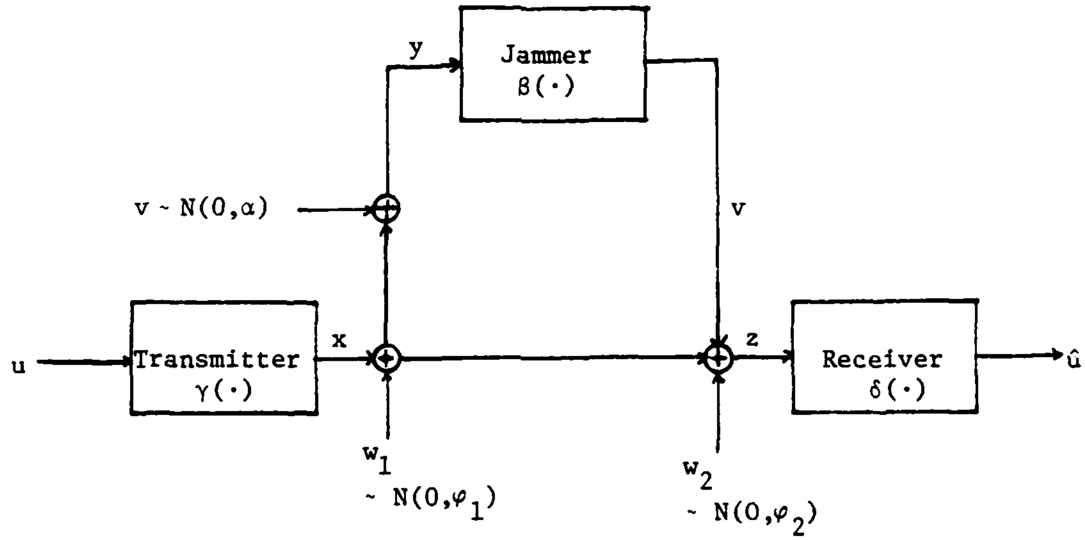

Figure 1.2. The Gaussian test channel with an intelligent jammer treated in [3].

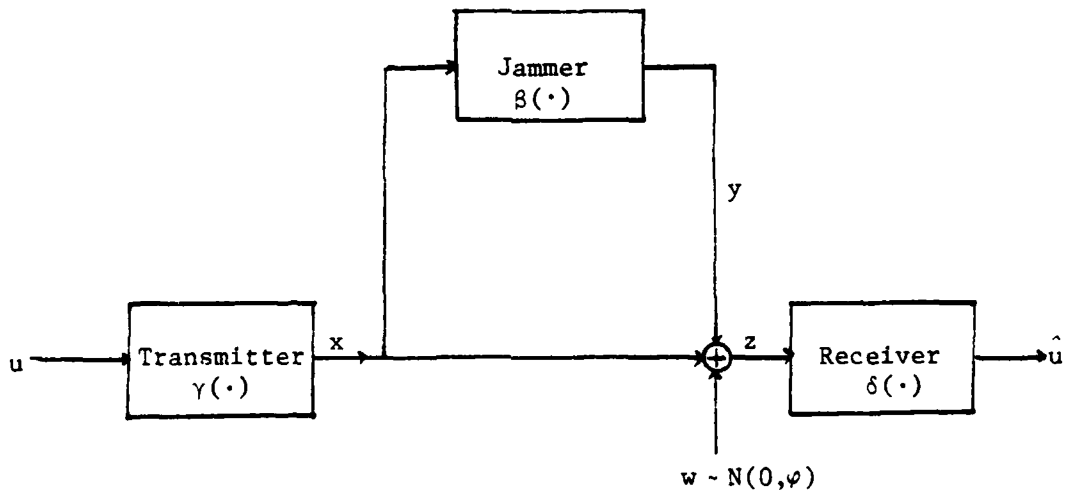

Figure 1.3. A simplified version of Figure 1.2.

1.4. Vector Extension of Cases (i) and (ii)

The vector version of case (i) is completely described by the communication system depicted in Fig. 1.4. Here, the input vector, $u \in \mathbb{R}^n$, is a sequence of independent, identically distributed Gaussian random vectors with mean zero and covariance matrix $\Sigma > 0$. An admissible encoder policy is a probability distribution on the set of all mxn linear mappings, $A : \mathbb{R}^n \to \mathbb{R}^m$, with second moment no greater than P. We denote the class of all such probability distributions by Ω_s. Further let $\Omega = \{A : Tr(A\Sigma A') \leq P\}$ and $z \in \mathbb{R}^m$ be a random vector such that

$$z = x + v + r$$

Here, the random vector, $v \in \mathbb{R}^m$, is controlled by an intelligent jammer, and is allowed to be correlated with the input vector, u. Denote the class of associated probability measures μ for v satisfying the power constraint $E(v'v) = Tr[\Lambda] \leq Q$, by ν. (Here Λ denotes the covariance matrix of v.) The channel noise, r, is a sequence of independent, identically distributed Gaussian random vectors with mean zero and covariance matrix, $R > 0$, and independent of v and u.

Finally, the estimator of the input vector u based on the measurement, z, will be denoted $\delta(\cdot)$, and the class of all such estimators will be denoted by D. There is a side channel between the encoder and the decoder, which provides the decoder with the structural information regarding the choice of the encoder.

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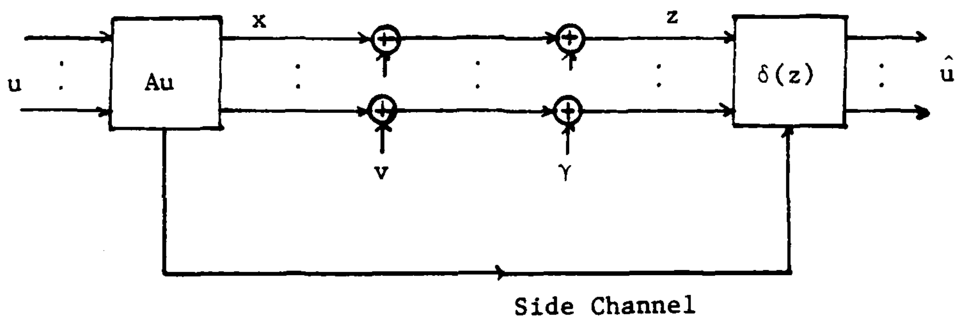

Figure 1.4. A schematic description of communication system for Vector Case (i).

Vector case (ii) is formulated in exactly the same way as vector case (i) described above, with the only two differences being that now there is no side channel and no probabilistic coding is allowed at the encoder. In vector case (ii) the encoding policy will be restricted to the class of linear decision rules, $X = Au$, where $A \in \Omega$ is deterministic. If, in the linear class, A is allowed to be random, no improvement can be achieved in the performance of the encoder-decoder pair. For vector case (i), the distortion measure is given by

$$J(\gamma,\delta,u) = E\{(\delta(x + v + r) - u)'(\delta(x + v + r) - u)\} \qquad (1.8)$$

and for case (ii) by

$$J(A,\delta,\mu) = E\{(\delta(Au + V + r) - u)'(\delta(Au + V + r) - u)\} \qquad (1.9)$$

We again seek a saddle-point solution for J, or worst-case solutions by min.max. and max.min. approaches if a saddle-point solution does not exist. These are conceptually the same as those introduced earlier for scalar versions and hence will not be discussed in great detail. We only point out, for future reference, that (1.7) can be rewritten as case (i):

$$J(\gamma^*,\delta^*,\mu) \le J(\gamma^*,\delta^*,\mu^*) \le J(\gamma,\delta,\mu^*) \ , \ \forall\gamma \in \Omega_s, \ \delta \in D, \ \mu \in \nu \qquad (1.10)$$

and case (ii):

$$J(A^*,\delta^*,\mu) \le J(A^*,\delta^*,\mu^*) \le J(A,\delta,\mu^*) \ , \ \forall A \in \Omega, \ \delta \in D, \ \mu \in \nu \qquad (1.11)$$

Lemma 1.1 is still valid for the vector cases, by defining $\bar{J}^*$ and $\underline{J}^*$ in a way similar to (1.1) and (1.4).

CHAPTER 2

SOLUTION FOR THE SCALAR COMMUNICATION SYSTEM

This chapter is concerned mainly with the derivation of solutions to all the scalar cases previously introduced in Chapter 1. First, the saddle-point solution for case (iii) is reviewed in Section 2.1. Following that, Section 2.2 provides the saddle-point solution for case (i) and its proof. Then, the min.max. solution and max.min. solution are presented and verified respectively, in Section 2.3 and Section 2.4. After presenting the solutions, we summarize and compare these results in the last section (Section 2.5).

2.1. Saddle-Point Solution for Case (iii)

In this section, we present (in Theorem 2.1) the saddle-point solution for the communication system of Fig. 1.3. Before doing this we first introduce some parametric regions in terms of k, c, and φ_1 which will be used throughout the chapter:

$$\begin{aligned} R_1 &: k \geq c \\ R_2 &: k < c \\ R_3 &: k^2 - ck + \varphi > 0 \\ R_4 &: k^2 - ck + \varphi \leq 0 \end{aligned} \tag{2.1}$$

Theorem 2.1 [3]: The communication system shown in Fig. 1.3 admits two saddle-point solutions $(\gamma^*, \delta^*, \beta^*)$ and $(-\gamma^*, -\delta^*, \beta^*)$ over the space $\Gamma_e \times \Gamma_d \times \Gamma_j$, where

$$\text{i)}\quad \gamma^*(u) = \begin{cases} \text{arbitrary} & \text{in } R_1 \\ cu & \text{in } R_2 \end{cases} \tag{2.2}$$

$$\text{ii)}\quad \beta^*(x) = \begin{cases} -x & \text{in } R_1 \\ -\dfrac{k}{c}\,x & \text{in } R_2 \cap R_3 \\ -\dfrac{(k^2+\varphi)}{c^2}\,x + \eta & \text{in } R_2 \cap R_4 \end{cases} \tag{2.3}$$

where η is a zero-mean Gaussian random variable with variance $k^2 - \dfrac{(k^2+\varphi)^2}{c^2}$ and independent of other random variables.

$$\text{iii)}\quad \delta^*(z) = \begin{cases} 0 & \text{in } R_1 \\ \dfrac{c-k}{(c-k)^2+\varphi}\,z & \text{in } R_2 \cap R_3 \\ \dfrac{z}{c} & \text{in } R_2 \cap R_4 \end{cases} \tag{2.4}$$

iv) the optimal value of J:

$$J^*(\gamma^*,\delta^*,\beta^*) = \begin{cases} 1 & \text{in } R_1 \\ \dfrac{\varphi}{(c-k)^2+\varphi} & \text{in } R_2 \cap R_3 \\ \dfrac{k^2+\varphi}{c^2} & \text{in } R_2 \cap R_4 \end{cases} \tag{2.5}$$

2.2. Saddle-Point Solution for Case (i)

The communication system of case (i) has been redepicted in Fig. 2.1. In the following, we will first summarize the saddle-point solution in Theorem 2.2, its necessary conditions in Corollary 2.1, and then present their proofs. At this point, we digress to clarify some notation. The pair

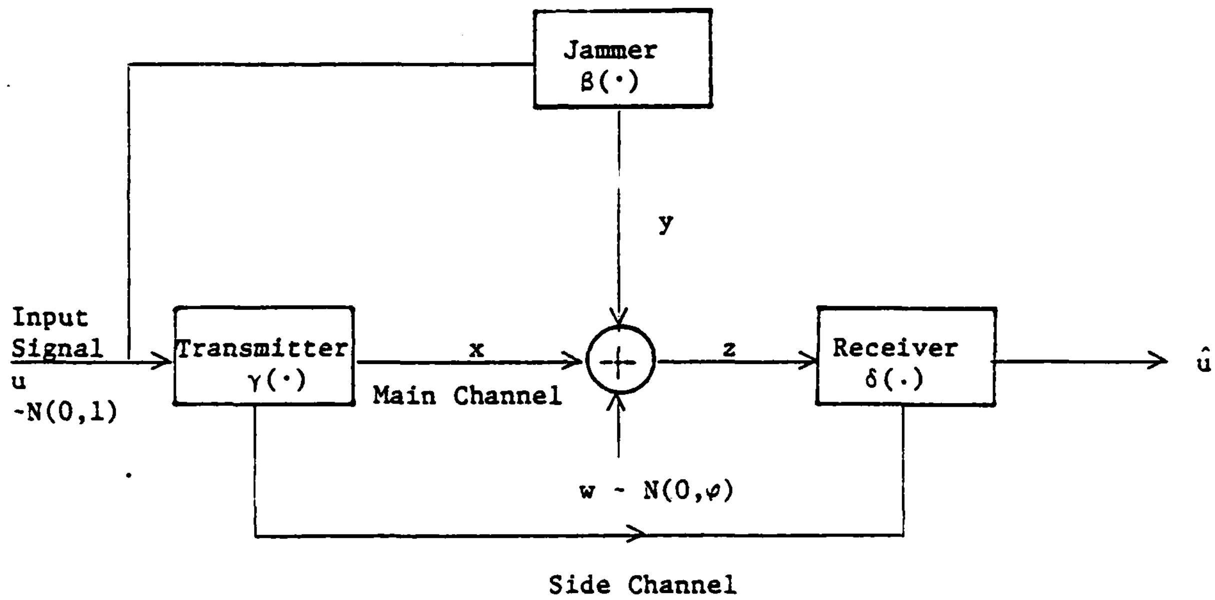

Figure 2.1. The Gaussian test channel for version (i).

(γ_1,δ_1) will mean that while the transmitter is using a certain encoding policy, γ, (either deterministic or a particular realization of a mixed policy), he informs the receiver of his policy in current use through the side channel, and in response the receiver decodes its input, z using the corresending decoding policy, δ_1. Furthermore, "w.p." will stand for "with probability." Hence, when the expression "(γ_1,δ_1) w.p. p_1" appears, this will mean that both the encoding policy, γ_1, and the decoding policy, δ_1, are used simultaneously with probability p_1.

Theorem 2.2: For the communication system of Fig. 2.1, there exists a saddle-point solution $(\gamma^*,\delta^*,\beta^*)$, given as follows:

$$\text{i)}\quad (\gamma^*,\delta^*) = \begin{cases} \left(cu, \dfrac{cz}{c^2+k^2+\varphi}\right) & \text{w.p.} 1/2 \\ \left(-cu, -\dfrac{cz}{c^2+k^2+\varphi}\right) & \text{w.p.} 1/2 \end{cases} \tag{2.6}$$

$$\text{ii)}\quad \beta^*(u) = \eta \tag{2.7}$$

where η is a zero-mean Gaussian random variable with variance k^2 and is independent of both u and w.

iii) The saddle-point value of J is

$$J^* = \frac{k^2+\varphi}{c^2+k^2+\varphi} \tag{2.8}$$

Corollary 2.1: (γ^*,δ^*) given in (2.6) is a unique saddle-point soluti n in the space $\Gamma_e \times \Gamma_d$. And if $\tilde{\beta}$ is a saddle-point solution in the space Γ_1, the jamming variable, y, generated by $\tilde{\beta}(\cdot)$, must be uncorrelated with the input, u, i.e., $E(u\ y) = 0$.

Proof for Theorem 2.2: We need to prove that the solution given in Theorem 2.2 satisfies the saddle-point inequality for $\forall\gamma \in \Gamma_e$, $\delta \in \Gamma_d$, $\beta \in \Gamma_j$

$$J(\gamma^*,\delta^*,\beta) \leq J(\gamma^*,\delta^*,\beta^*) \leq J(\gamma,\delta,\beta^*) \tag{2.9}$$

The proof will be completed in two steps. First, we will show that the right-hand side inequality of (2.9) is satisfied, i.e., the pair of encoding and decoding policies, (γ^*,δ^*), as specified by (2.6), minimize $J(\gamma,\delta,\beta^*)$ over the space $\Gamma_e \times \Gamma_d$, given that the jamming policy, β^*, is in current use by the jammer. Secondly, the left-hand side inequality of (2.9) will be verified, i.e., when the pair of encoding and decoding policies, (γ^*,δ^*), is chosen by the transmitter and the receiver, $J(\gamma^*,\delta^*,\beta)$ is maximized by the jamming policy, β^*, over the space Γ_j.

a) Verification of the right-hand side inequality: Assume that the jamming policy, β^*, is given by (2.7). Hence, the input signal to the receiver is

$$z = x + \eta + w$$

Referring to [1], for a Gaussian test channel, the transmitter should amplify the magnitude of the input, u, linearly at his given energy level. Thus,

$$\gamma(u) = cu \quad \text{or} \quad -cu$$

In cooperation with the transmitter, the receiver must adapt Bayes' estimate correspondingly in either encoding policies. When $\gamma^*(u) = cu$, the Bayes' estimate is

$$\delta(z) = \bar{u} + \frac{E[(u - \bar{u})(z - \bar{z})]}{E[(z - \bar{z})^2]}(z - \bar{z}) \quad ; \quad \bar{u} \triangleq E[u]$$

because, the sum of $\eta + w$ is still Gaussian, $Eu = 0$ and $E(z) = E(x) + E\eta + Ew = 0$.

Hence, $\delta(z) = \dfrac{E(uz)}{E(z^2)} z$

$$E(z^2) = E(cu + \eta + w)^2 = c^2 + k^2 + \varphi$$

$$E(uz) = E(u(cu + \eta + w)) = cEu^2 = c$$

$\delta(z)$ becomes

$$\delta^*(z) = \frac{c}{c^2 + k^2 + \varphi} z$$

Similarly, for $\delta^*(u) = -cu$, we obtain

$$\delta^*(z) = \frac{-c}{c^2 + k^2 + \varphi} z$$

So, this proves the validity of (2.6) in Theorem 2.2.

b) <u>Verification of the left-hand side inequality</u>: Assuming that the pair of encoding and decoding policies (γ^*, δ^*) are as given in (2.6), we first compute the expected mean square error, $J(\gamma^*, \delta^*, \beta)$ by conditioning on the specific structural realization of encoding-decoding policies in current use.

i) When $(cu, \dfrac{cz}{c^2 + k^2 + \varphi})$ is the realized policy pair, we have

$$J_{11} = E\left(\frac{cz}{c^2 + k^2 + \varphi} - u\right)^2$$

$$= E\left(\frac{c}{c^2 + k^2 + \varphi}(cu + y + w) - u\right)^2 \tag{2.10}$$

$$J_{11} = \frac{(k^2 + \varphi)^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2}{(c^2 + k^2 + \varphi)^2} E(y^2) + \frac{c^2}{(c^2 + k^2 + \varphi)^2}$$
$$- \frac{2c(k^2 + \varphi)}{(c^2 + k^2 + \varphi)^2} E(uy)$$

ii) with $(u) = -cu$, and $\delta^* = -\dfrac{cz}{c^2 + k^2 + \varphi}$, we obtain

$$J_{12} = E\left(-\frac{cz}{c^2 + k^2 + \varphi} - u\right)^2$$

$$= E\left(-\frac{c}{c^2 + k^2 + \varphi}(-cu + y + w) - u\right)^2 \tag{2.11}$$

$$J_{12} = \frac{(k^2 + \varphi)^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2}{(c^2 + k^2 + \varphi)^2} E(y^2) + \frac{c^2}{(c^2 + k^2 + \varphi)^2}\varphi + \frac{2c(k^2 + \varphi)}{(c^2 + k^2 + \varphi)^2} E(uy)$$

By unconditioning the above conditional values of J, we obtain

$$J = E(\delta^*(w - u)^2 = \frac{1}{2}J_{11} + \frac{1}{2}J_{12} = \frac{(k^2 + \varphi)^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2}{(c^2 + k^2 + \varphi)^2} E(y^2) + \frac{c^2}{(c^2 + k^2 + \varphi)^2} \tag{2.12}$$

which indicates that J is dependent only on the second order moment of y. Hence, the maximizing solution is any random variable with second order moment equal to k^2; and the Gaussian random variable η, with mean zero and variance k^2 is one such random variable.

Hence, $$J^* = \frac{(k^2 + \varphi)^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2k^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2\varphi}{(c^2 + k^2 + \varphi)^2} = \frac{k^2 + \varphi}{c^2 + k^2 + \varphi}$$

This then completes the proof of Theorem 2.2. □

Proof for Corollary 2.1: By the property of interchangeability of the saddle-point solution [2], any further possible saddle-point solution

(besides (2.7)) for the jamming policy should be in equilibrium with (γ^*, δ^*) given in (2.6). We will now verify the uncorrelation between the input, u, and the jamming variable, y, by contradiction.

If $E(uy) > 0$, choosing

$$(\hat{\gamma}, \hat{\delta}) = (cu, \frac{cz}{c^2 + k^2 + \varphi}) \quad \text{w.p. } 1 \tag{2.13}$$

then it follows from (2.10) and (2.12) that $(\hat{\gamma}, \hat{\delta})$ will yield lower value for J than the (γ^*, δ^*) in (2.6) does. This contradicts the optimality of (γ^*, δ^*) given in (2.6).

An analogous reasoning can be made for the case when $E(uy) < 0$ with (2.11) and (2.12). Hence, this proves that $E(uy) = 0$ is necessary for $\tilde{\beta}$ to be a saddle-point solution. Next, we show that (γ^*, δ^*) in (2.6) is unique in the space $\Gamma_e \times \Gamma_d$.

Assume

$$(\hat{\gamma}, \hat{\delta}) = \begin{cases} (cu, \frac{cz}{c^2 + k^2 + \varphi}) & \text{w.p. } p \\ (-cu, -\frac{cz}{c^2 + k^2 + \varphi}) & \text{w.p. } 1-p \end{cases} \tag{2.14}$$

Again, by the property of interchangability of the saddle-point solutions, if $(\tilde{\gamma}, \tilde{\delta})$ in (2.14) is a saddle-point solution, then it will be in equilibrium with β^* in (2.7). So β^* in (2.7) will maximize J given that $(\hat{\gamma}, \hat{\delta})$ in (2.14) is adopted by the transmitter and the receiver.

By (2.10) and (2.11),

$$\begin{aligned} J &= pJ_{12} + (1 - p)J_{12} \\ &= \frac{(k^2 + \varphi)^2}{(c^2 + k^2 + \varphi)^2} + \frac{c^2}{(c^2 + k^2 + \varphi)^2} E(y^2) + \frac{c^2}{(c^2 + k^2 + \varphi)^2} \varphi \\ &\quad - \frac{2c(k^2 + \varphi)}{(c^2 + k^2 + \varphi)^2} (1 - 2p)E(uy) \end{aligned} \tag{2.15}$$

By inspection of (2.15), since β^* in (2.7) is the maximum solution, the last term in (2.15) must be zero, i.e., $1 - 2p = 0 \Rightarrow p = 1/2$, so $p = 1/2$ is the only solution for $(\hat{\gamma},\hat{\delta})$ to be in equilibrium with β^* in (2.7). Therefore, the uniqueness of (γ^*,δ^*) in (2.6) has been proved. □

The following will now provide an introduction to the analyses of the remaining sections, where we highlight the motivation to derive the worst-case solutions by min.max. and max.min. approaches in case (ii). Let us first compare the max.min. approaches in cases (i) and (ii). Given a specific jamming policy, β^*, in both cases the transmitter and the receiver will achieve the same optimal value of J since there would be no need to randomize encoder policies and hence to use the side channel. Thus, the max.min. approaches would lead to the same lower value of the game in both cases. On the other hand, the min.max. approaches will yield the same upper value, $\bar{J}^*$ in cases (ii) and (iii) due to the fact that any randomization in the encoder policy is not helpful in minimizing J and, hence, both jammers in cases (ii) and (iii) will access the same information from the given (γ,δ).

It then follows from the above discussion and the statements of Theorems 2.1 and 2.2 that in case (ii) the min.max. value of J, $\bar{J}^*$, is

always strictly greater than the max.min. value of J, $\underline{J}^*$, and, therefore, by Lemma 1.1, a saddle-point solution will not exist in case (ii).

2.3. Derivation of a Min.Max. Solution for Case (ii)

Now, we eliminate the side channel in the communication system of Fig. 2.1. The result is the communication system depicted in Fig. 2.2, for which the min.max. solution is given below as Theorem 2.3.

Theorem 2.3: For the scalar problem of case (ii), there exist two min.max. solutions: $(\gamma^*,\delta^*,\beta^*)$ and $(-\gamma^*,-\delta^*,-\beta^*)$ where

$$\text{i)}\quad \gamma^*(u) = \begin{cases} \text{arbitrary} & \text{in } R_1 \\ cu & \text{in } R_2 \end{cases} \tag{2.16}$$

$$\text{ii)}\quad \beta^*(u) = \begin{cases} \gamma^*(-u) & \text{in } R_1 \\ -ku & \text{in } R_2 \cap R_3 \\ \text{any arbitrary second-order random variable with } E(v^2) = k^2 & \text{in } R_2 \cap R_4 \end{cases} \tag{2.17}$$

$$\text{iii)}\quad \delta^*(z) = \begin{cases} 0 & \text{in } R_1 \\ \dfrac{c-k}{(c-k)^2+\varphi}\, z & \text{in } R_2 \cap R_3 \\ \dfrac{z}{c} & \text{in } R_2 \cap R_4 \end{cases} \tag{2.18}$$

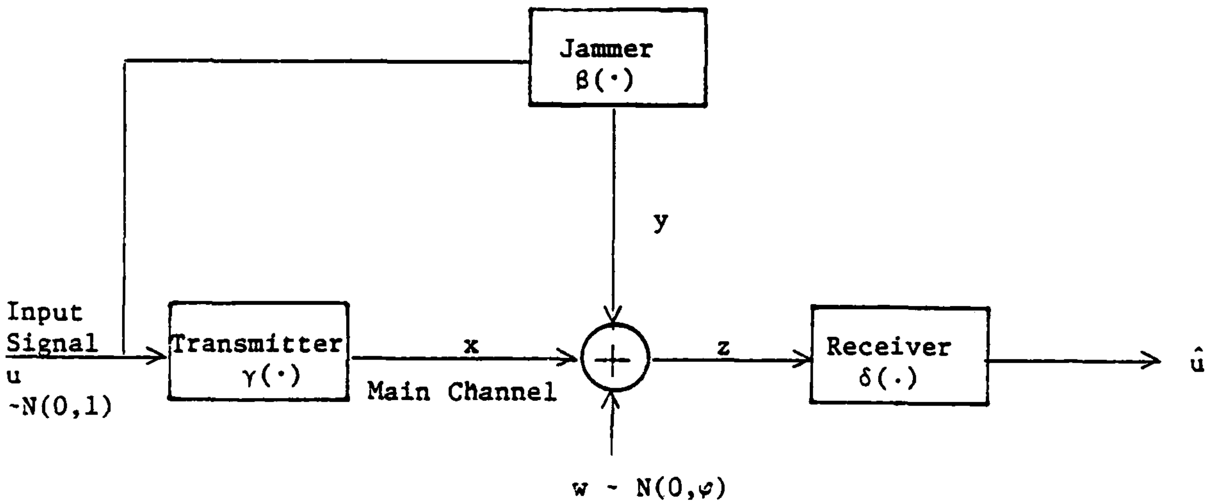

Figure 2.2. The Gaussian test channel for version (ii).

$$\text{iv)}\quad \bar{J}^* = \begin{cases} 1 & \text{in } R_1 \\ \dfrac{\varphi}{(c-k)^2+\varphi} & \text{in } R_2 \,\Omega\, R_3 \\ \dfrac{k^2+\varphi}{c^2} & \text{in } R_2 \,\Omega\, R_4 \end{cases} \qquad (2.19)$$

Proof: In the following, we first solve the minimization problem given that the jamming policy, $\beta(\cdot)$, is always specified in terms of the encoding policy, $\gamma(u)$ or x. Of course, the minimum value obtained under this specified jamming policy will constitute a lower bound for the original min.max. problem of case (ii). Later, we will prove that this lower bound can actually be achieved by some encoding and decoding policies.

Region R_1: In mathematical terms, we have

$$\bar{J}^* = \min_{\substack{\gamma\in\Gamma_e \\ \delta\in\Gamma_d}} \max_{\beta\in\Gamma_j} J \geq \min_{\substack{\gamma\in\Gamma_e \\ \delta\in\Gamma_d}} \Gamma(\gamma,\delta,\beta(u) = \gamma(-u)) \qquad (2.20)$$

Consider the right-hand side of (2.20): $\beta(u) = \gamma(-u)$ means that the jammer is adopting the same policy of the encoder to generate the jamming variable, y but using the negative value of the current input, u as its argument (he can do that because $k \geq c$).

Hence, x and y are conditionally independent, given u. With $z = x + y + w$, the conditional density of u given z will be even symmetric: i.e., $f(u|z) = f(-u|z)$.

Therefore, the Bayes' estimate will be zero,

$$\delta(z) = E(u|z) = \int_{-\infty}^{\infty} uf(u|z)du = 0$$

$$J = E(0 - u)^2 = Eu^2 = 1$$

From (2.20),

$$\bar{J}^* \geq 1 \tag{2.21}$$

This lower bound in (2.21) has been attained by γ^* and δ^* in (2.16) and (2.18), correspondingly.

Region $R_2 \cap R_3$: Similar to Region R_1, we have

$$\min_{\substack{\gamma\in\Gamma_e \\ \delta\in\Gamma_d}} \max_{\beta\in\Gamma_j} J \geq \min_{\substack{\gamma\in\Gamma_e \\ \delta\in\Gamma_d}} J(\gamma,\delta,\beta(u) = -\frac{k}{c} E(x|u)) \tag{2.22}$$

Considering the right-hand side of (2.22), the input signal to the receiver is:

$$\begin{aligned} z &= x - \frac{k}{c} E(x|u) + w \\ &= (1 - \frac{k}{c})x + \frac{k}{c}(x - E(x|u)) + w \end{aligned} \tag{2.23}$$

Let us digress to consider the following two optimization problems,

(a) our original problem: Seeking to send the input u optimally through the following channel in the sense of mean square error,

$$z = (1 - \frac{k}{c})x + \frac{k}{c}(x - E(x|u)) + w$$

Here x is generated by the encoding policy, $\gamma(u)$, satisfying the power constraint $E(x^2) \leq c^2$, and $w \sim N(0,\varphi)$.

(b) the transformed problem split expression (2.23) into two parts: z_1 and z_2, where

$$\begin{cases} z_1 = (1 - \frac{k}{c})x + r \\ z_2 = \frac{k}{c}(x - E(x|u)) \end{cases} \tag{2.24a}$$

Instead of z in (2.23), z_1 and z_2 in (2.24a) are received by the receiver. This transformed problem achieves a minimum square error which is lower than the one of the problem (a). By manipulating on these data z_1 and z_2 algebraically, we obtain

$$\begin{cases} \hat{z}_1 \equiv \dfrac{z_1}{1 - \frac{k}{c}} = x + \dfrac{1}{1 - \frac{k}{c}} r \\ \hat{z}_2 \equiv \dfrac{z_2}{\frac{k}{c}} = x - E(x|u) \end{cases} \tag{2.24b}$$

Further,

$$\begin{cases} \hat{z}_1 = x + \dfrac{1}{1 - \frac{k}{c}} r \\ \hat{z}_1 - \hat{z}_2 = E(x|u) + \dfrac{1}{1 - \frac{k}{c}} r \end{cases} \tag{2.24c}$$

Since $\hat{z}_1$ and $\hat{z}_1 - \hat{z}_2$ are obtained from z_1 and z_2 by nonsingular linear transformations, we see from (2.24c) that the randomized encoding policy is totally unnecessary; hence, we can say that an optimal encoding policy exists as a deterministic function of u. When the encoding policy is restricted to deterministic rules, z_2 becomes zero in (2.24a).

The channel in the transformed problem (b) is thus reduced to single Gaussian test channel. Hence, $x = cu$ or $-cu$ is an optimal encoding policy in (b). Without loss of generality, we choose $x = cu$, hence, (2.24a) or (2.23) becomes

$$z = (c - k)u + w$$

and
$$\delta(z) = E(u|z) = \frac{c - k}{(c - k)^2 + \varphi} z \tag{2.25}$$

and
$$J = E(u - \delta(z))^2 = \frac{\varphi}{(c - k)^2 + \varphi}$$

(2.22) becomes

$$\min_{\substack{\gamma \in \Gamma_e \\ \delta \in \Gamma_d}} \; \max_{\beta \in \Gamma_j} \; J \geq \frac{\varphi}{(c - k)^2 + \varphi} \tag{2.26}$$

The next thing to be proved is that (2.26) is in fact an equality, given γ^* and δ^* in (2.16) and (2.18), respectively. Then,

$$J = E(\delta(z) - u)^2 = E\left(\frac{c - k}{(c - k)^2 + \varphi}(cu + y + w) - u\right)^2$$

$$= \frac{(k(c - k) + \varphi)^2}{((c - k)^2 + \varphi)^2} + \frac{(c - k)^2 \varphi}{((c - k)^2 + \varphi)^2} + \frac{(c - k)^2}{((c - k)^2 + \varphi)^2} E(y^2)$$

$$+ 2\frac{((c - k)(k(c - k) - \varphi)}{((c - k)^2 + \varphi)^2} E(uy) \tag{2.27}$$

The maximum of J above with respect to β can be achieved by any policy β satisfying

$$E(y^2) = k^2$$

and $$E(uy) = -k \qquad \text{since } ck - k^2 - \varphi < 0 \quad .$$

By Cauchy-Schwartz inequality, $E(\mu\beta(u)) = -k$ can be achieved by choosing $\beta^*(u) = -ku$, uniquely. Substituting $E(\beta^2(u)) = k^2$ and $E(u\beta(u)) = -k$ back into (2.27), we obtain

$$J^* = \frac{\varphi}{\varphi + (c-k)^2} \tag{2.28}$$

The value of J obtained above is the same as that in (2.26). Hence, the proof has been completed for the region $R_2 \cap R_3$.

Region $R_2 \cap R_4$: Following the lines of the previous derivation, we have

$$\min_{\substack{\gamma \in \Gamma_e \\ \delta \in \Gamma_d}} \; \max_{\beta \in \Gamma_j} \; J \geq \min_{\substack{\gamma \in \Gamma_e \\ \delta \in \Gamma_d}} \; J(\gamma, \delta, \beta(u) = -\frac{(k^2+\varphi)}{c^2} E(x|u) + \eta) \tag{2.29}$$

where $\eta \sim N(0, k^2 - \frac{(k^2+\varphi)^2}{c^2})$

For the right-hand side of (2.29), write z as

$$z = x - \frac{k^2+\varphi}{c^2} E(x|u) + \eta + w$$

$$= (1 - \frac{k^2+\varphi}{c^2})x - \frac{k^2+\varphi}{c^2}(x - E(x|u)) + r \tag{2.30}$$

where $$r = \eta + w \quad N(0, k^2 + \varphi - \frac{(k^2+\varphi)^2}{c^2})$$

Similar to the procedure in the Region $R_2 \cap R_3$, again, the optimal encoding policy will be

$$\gamma(u) = cu \text{ or } -cu \tag{2.31}$$

Without loss of generality, we choose $\gamma(u) = cu$. Then, (2.30) becomes

$$z = \left(1 - \frac{k^2 + \varphi}{c^2}\right) cu + \gamma \tag{2.32}$$

so,

$$\delta(z) = E(u|z) = \frac{\left(1 - \frac{k^2 + \varphi}{c^2}\right) c}{\left(1 - \frac{k^2 + \varphi}{c^2}\right)^2 c^2 + k^2 - \frac{(k^2 + \varphi)^2}{c^2} + \varphi} z$$

$$= \frac{z}{c} \tag{2.33}$$

Then,

$$J = \frac{k^2 + \varphi}{c^2} \tag{2.34a}$$

and

$$\min_{\substack{\gamma \in \Gamma_e \\ \delta \in \Gamma_d}} \ \max_{\beta \in \Gamma_j} \ J \gtrsim \frac{k^2 + \varphi}{c^2} \tag{2.34b}$$

To prove attainability of the lower bound in (2.34b), take γ^* and δ^* as given in (2.16) and (2.18), respectively,

$$J = E(\gamma^*(z) - u)^2 = \frac{1}{c^2} (\varphi + E(y^2)) \tag{2.35}$$

Maximizing J above with respect to admissible β, the solution is any β satisfying $E(y^2) = k^2$, which in turn leads to

$$J^* = \frac{1}{c^2} (\varphi + k^2) \tag{2.36}$$

Again, the value of J in (2.36) is the same as the lower bound in (2.34b). This then completes the proof of Theorem 2.3. □

2.4. Verification of a Max.Min. Solution for Case (ii)

After the completion of the solution of the min.max. problem, we now proceed with the max.min. solution for case (ii), which is given in the following theorem.

Theorem 2.4: For the scalar problem of case (ii), there exist two max.min. solutions over $\Gamma_e \times \Gamma_d \times \Gamma_j$: $(\gamma^*, \delta^*, \beta^*)$ and $(-\gamma^*, -\delta^*, -\beta^*)$ where

$$\text{i)} \quad \gamma^*(u) = cu \tag{2.37}$$

$$\text{ii)} \quad \beta^*(u) = \eta \tag{2.38}$$

where η is the Gaussian random variable with $E(u\eta) = 0$ and $E(\eta^2) = k^2$

$$\text{iii)} \quad \delta^*(z) = \frac{cz}{c^2 + k^2 + \varphi} \tag{2.39}$$

and

$$\text{iv)} \quad \underline{J}^* = \frac{k^2 + \varphi}{c^2 + k^2 + \varphi} \tag{2.40}$$

Proof: In the following we first solve the max.min. problem given that the encoding and decoding policies can only be linear decision rules. Of course, the max.min. value obtained under this restriction on the spaces of encoding and decoding policies will constitute an upper bound for the original max.min. problem of case (ii). Later, we will prove that this upper bound can actually be achieved at a certain jamming policy. In mathematical terms, we have

$$\max_{\beta \in \Gamma_j} \ \min_{\substack{\gamma \in \Gamma_e \\ \delta \in \Gamma_d}} J \leq \max_{\beta \in \Gamma_j} \ \min_{\substack{\alpha, \Delta \in \mathbb{R} \\ \gamma(u) = \alpha u \\ \delta(z) = \Delta z \\ |\alpha| \leq c}} J \tag{2.41}$$

where on the right-hand side of the inequality, (2.41), the minimization is carried out over the real scalars, α,Δ.

Starting with the right-hand side,

$$J = E(\delta - u)^2 = E(\Delta(\alpha u + y + w) - u)^2$$

$$= E[(\Delta\alpha - 1)u + \Delta y + \Delta w]^2$$

$$\Rightarrow \quad J = (\Delta\alpha - 1)^2 + \Delta^2\varphi + \Delta^2 E(y^2) + 2\Delta(\Delta\alpha - 1)E(uy) \tag{2.42}$$

It is intuitively clear that $E(y^2) = k^2$ will be chosen by the jammer (provided that $\Delta^2 > 0$), so that

$$J = (\Delta\alpha - 1)^2 + \Delta^2\varphi + \Delta^2 k^2 + 2\Delta(\Delta\alpha - 1)E(uy) \tag{2.43}$$

Differentiating J with respect to Δ:

$$\frac{\partial}{\partial\Delta} J = 2\alpha(\Delta\alpha - 1) + 2\Delta(\varphi + k^2) + 2(2\alpha\Delta - 1)E(uy) \tag{2.44}$$

$$\frac{\partial^2}{\partial\Delta^2} J = 2\alpha^2 + 2\varphi + 2k^2 + 4\alpha E(uy) > 0 \tag{2.45}$$

and setting $\frac{\partial}{\partial\Delta} J\Big|_{\Delta^*} = 0$

$$\Rightarrow \alpha(\Delta^*\alpha - 1) + \Delta^*(\varphi + k^2) + (2\alpha\Delta^* - 1)E(uy) = 0$$

we obtain

$$\Delta^* = \frac{\alpha + E(uy)}{\alpha^2 + k^2 + \varphi + 2\alpha E(uy)} \tag{2.46}$$

as the unique minimizing Δ^*, for fixed α, since the second derivative is positive by (2.45).

Substituting Δ^* back into J, we obtain

$$J\Big|_{\Delta^*} = \frac{k^2 + \varphi E^2(uy)}{\alpha^2 + k^2 + \varphi + 2\alpha E(uy)} \tag{2.47}$$

and, minimizing $J\big|_{\Delta^*}$ with respect to α, yields

$$\alpha^* = \begin{cases} c & \text{if } E(uy) > 0 \\ -c & \text{if } E(uy) < 0 \\ c \text{ or } -c & \text{if } E(uy) = 0 \end{cases} \tag{2.48}$$

for simplicity, letting V = E(uy), and substituting α^* back into $J\big|_{\Delta^*}$ of (2.47), we obtain $J\big|_{\Delta^*,\alpha^*}$ as follows:

$$J\Big|_{\Delta^*,\alpha^*} = \begin{cases} \dfrac{k^2 + \varphi - v^2}{c^2 + k^2 + \varphi - 2cv} & \text{if } v \le 0 \\[2ex] \dfrac{k^2 + \varphi - v^2}{c^2 + k^2 + \varphi + 2cv} & \text{if } v \ge 0 \end{cases} \tag{2.49}$$

Since $J\big|_{\Delta^*,\alpha^*}$ depends only on the correlation coefficient of the input signal, u, and the jamming variable, y, we simply need to maximize $J\big|_{\Delta^*,\alpha^*}$ with respect to the scalar, V. Differentiating $J\big|_{\Delta^*,\alpha^*}$ with respect to v, we obtain

$$\frac{dJ\big|_{\Delta^*,\alpha^*}}{dv} = \begin{cases} -\dfrac{2(v - c)(k^2 + \varphi - cv)}{(k^2 + \varphi + c^2 - 2cv)^2} > 0 & \text{if } v < 0 \\[2ex] -\dfrac{2(v + c)(k^2 + \varphi + cv)}{(k^2 + \varphi + c^2 + 2cv)} < 0 & \text{if } v > 0 \end{cases} \tag{2.50}$$

where $J\Big|_{\Delta^*,\alpha^*}(0) = \frac{k^2 + \varphi}{c^2 + k^2 + \varphi}$ and $\frac{dJ\Big|_{\Delta^*,\alpha^*}}{dv}\Bigg|_{v = 0}$ is undefined.

Hence, $\arg (\max_v J\Big|_{\Delta^*,\alpha^*}) = 0$ and then $E(uy) = 0$. Substituting $E(uy) = 0$ back into (2.46) and (2.48) giving Δ^* and α^*, respectively, we obtain

$$\alpha^* = c$$
$$\Delta^* = \frac{c}{c^2 + k^2 + \varphi} \tag{2.51}$$

and β^* is any second-order random variable such that

$$E(uy) = 0 \quad \text{and} \quad E(y^2) = k^2 \tag{2.52}$$

So,

$$J^* = J\Big|_{\Delta^*,\alpha^*}(0) = \frac{k^2 + \varphi}{c^2 + k^2 + \varphi} \tag{2.53}$$

Now, we need to show that the inequality in (2.41) is in fact an equality for some jamming policy satisfying (2.52). Towards this end, we first note that a possible jamming policy satisfying (2.52) is the Gaussian random variable $\eta \sim N(0, k^2)$ independent of u and w. For this choice, the signal at the input of the receiver becomes

$$z = x + \eta + w$$

which is clearly the Gaussian test channel considered in [1]. We know from [1] that in the general class of endcoder-decoder mappings, the solution is linear and given by $\gamma^*(u) = cu$ and $\delta^*(z) = \frac{c}{c^2 + k^2 + \varphi}z$. The corresponding value of J is

$$J^* = \frac{k^2 + \varphi}{k^2 + c^2 + \varphi} \quad .$$

Since J^* above equals to $J\big|_{\Delta^*,\alpha^*}(0)$ in (2.53), it follows that the left-hand side of (2.41) cannot be strictly less than the right-hand side. Hence, the linear encoder-decoder policy pair obtained above is indeed optimal for the max.min. problem of case (ii).

This then completes the proof of Theorem 2.4. □

From Theorems 2.3 and 2.4, since in case (ii) the min.max. value of J, $\bar{J}^*$ is always strictly greater than the max.min. value of J, $\underline{J}^*$, the following corollary follows:

Corollary 2.2: In case (ii) a saddle-point solution does not exist. □

2.5. Summary of the Scalar Solutions in All Cases

Let us now collect the results presented in the previous sections and present them comparatively in Table 2.1. The highlights of this comparison are the following:

1) The saddle-point value of J in case (i) is the same as the max.min. value of J, $\underline{J}^*$ in case (ii). Notationally, max.min. of (i) $\Longleftrightarrow$ max.min. of (ii).

2) The min.max. value of J, $\bar{J}^*$ in case (ii) attains the saddle-point value of J in case (iii). Notationally, min.max. of case (ii) $\Longleftrightarrow$ min.max. of case (iii).

3) The value of $\underline{J}^*$ in case (ii) and the saddle-point value of J in case (i) are always strictly less than the min.max. value of J, $\bar{J}^*$ in case (ii) and the saddle-point value of J in case (iii), respectively.

TABLE 2.1. SUMMARY OF THE RESULTS IN ALL SCALAR PROBLEMS

Case (i) Saddle-Point Solution $(\gamma^*,\delta^*,\beta^*)$	Case (ii) Min.Max. Solution $(\gamma^*,\delta^*,\beta^*)$ & $(-\gamma^*,-\delta^*,-\beta^*)$	Case (ii) Max.Min. Solution $(\gamma^*,\delta^*,\beta^*)$ & $(-\gamma^*,-\delta^*,-\beta^*)$	Case (iii) Saddle-Point Solution $(\gamma^*,\delta^*,\beta^*)$ & $(-\gamma^*,-\delta^*,\beta^*)$
$(\gamma^*,\delta^*) = \begin{cases} (cu, \frac{cz}{c^2+k^2+\varphi}) & \text{w.p.}1/2 \\ (-cu, -\frac{cz}{c^2+k^2+\varphi}) & \text{w.p.}1/2 \end{cases}$ $\beta^* = \eta \sim N(0,k^2)$ $J^* = \frac{k^2+\varphi}{k^2+\varphi+c^2}$	R_1: $\gamma^*(u) =$ arbitrary $\beta^*(u) = \gamma^*(-u)$ $\delta^*(z) = 0$ $J^* = 1$	$\gamma^* = cu$ $\delta^* = \frac{c}{c^2+k^2+\varphi} z$ $\beta^* = \eta \sim N(0,k^2)$ $J^* = \frac{k^2+\varphi}{k^2+\varphi+c^2}$	R_1: $\gamma^* =$ arbitrary $\delta^*(z) = 0$ $\beta^* = -x$ $J^* = 1$
	$R_2 \cap R_3$: $\gamma^* = cu$ $\delta^*(z) = \frac{c-k}{(c-k)^2+\varphi} z$ $\beta^*(u) = -ku$ $J^* = \frac{\varphi}{(c-k)^2+\varphi}$		$R_2 \cap R_3$: $\gamma^*(u) = cu$ $\delta^*(z) = \frac{c-k}{(c-k)^2+\varphi} z$ $\beta^*(x) = -\frac{k}{c} x$ $J^* = \frac{\varphi}{(c-k)^2+\varphi}$
	$R_2 \cap R_4$: $\gamma^*(u) = cu$ $\delta^*(z) = \frac{z}{c}$ $\beta^*(u) = \eta$ s.t. $E(\eta^2) = k^2$ $J^* = \frac{k^2+\varphi}{c^2}$		$R_2 \cap R_4$: $\gamma^*(u) = cu$ $\delta^*(z) = \frac{z}{c}$ $\beta^*(x) = -\frac{k^2+\varphi}{c^2} x + w$ $J^* = \frac{k^2+\varphi}{c^2}$ where $w \sim N(0, k^2 - \frac{(k^2+\varphi)^2}{c^2})$

CHAPTER 3

SOLUTIONS FOR COMMUNICATION SYSTEMS WITH VECTOR SIGNALS AND VECTOR CHANNELS

This chapter is concerned with the derivation of saddle-point, min.max., and max.min. solutions for vector case (i) and (ii). In Section 3.1, the saddle-point solution for vector case (i) is derived and is summarized in Theorem 3.1. To gain a deeper insight into and to illustrate Theorem 3.1, some numerical examples are solved in Section 3.2. Following that, Section 3.3 provides a discussion of the derivation of a max.min. solution for vector case (ii), and summarizes the main results in Theorem 3.2. Then, the min.max. approach for vector case (ii) is discussed in Section 3.4. It seems that there do not exist closed form expressions for the min.max. solution (vector case (ii)) for all parametric regions; however, in some regions in the parameter space the solution can be obtained in closed-form and this has been presented in Theorem 3.3. The section also includes a numerical example to illustrate the theorem. Finally, the last section contains a comparative discussion of the results presented in earlier sections.

3.1. The Saddle-Point Solution for Vector Case (i)

Before presenting the saddle-point solution in Theorem 3.1, we first introduce some notation in the following.

Let T, U be unitary matrices that diagonalize R and Σ, respectively, and order their eigenvalues, i.e.,

$$\pi'R\pi = \text{diag}(r_1, r_2, \ldots, r_m) \;;\; 0 < r_1 \le r_2 \ldots \le r_m \tag{3.1}$$

and

$$U\Sigma U' = \text{diag}(\lambda_1, \ldots, \lambda_n) \;;\; \lambda_1 \ge \ldots \ge \lambda_n > 0 \tag{3.2}$$

Define an $h^* x h^*$ matrix T_o as

$$T_o = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \tag{3.3}$$

and let an mxm matrix H_i be introduced as

$$\begin{cases} H_o = I_m \\ H_i = \begin{pmatrix} T_o^i & 0 \\ 0 & I \end{pmatrix} \quad . \quad \text{for } i=1,\ldots,h^*-1 \end{cases} \tag{3.4}$$

Further, let

$$W = \pi \text{ diag } (\sigma, \ldots, \sigma, r_{h^*+1}, \ldots, r_m) \; \pi' = \pi \text{ diag}(w_1, \ldots, w_m) \; \pi' \tag{3.5}$$

and

$$\sigma = \frac{r_1 + \ldots + r_{h^*} + Q}{h^*} \tag{3.6}$$

where h^* is the smallest positive integer such that the following inequality holds for $h = h^*+1$:

$$(r_1 - r_1) + \ldots + (r_h - r_{h-1}) > Q \tag{3.7}$$

If there is no $h \le m$ for which the above inequality holds, then let $h^* = m$.

Define an mxn matrix $G = [g_{ij}]$ by

$$\begin{cases} g_{ii} = \left(\sqrt{\dfrac{\lambda_i w_i}{\alpha_o}} - w_i\right)^{1/2} / \sqrt{\lambda_i} & \text{for } i \le \ell \\ g_{ij} = 0 & \text{for } i \neq j, \text{ or } i > \ell \end{cases} \tag{3.8}$$

with

$$\sqrt{\alpha_o} = \frac{\sum_{i=1}^{\ell} \sqrt{\lambda_i w_i}}{P + \sum_{i=1}^{\ell} w_i} \tag{3.9}$$

and ℓ is the maximum integer which satisfies the following inequality:

$$\frac{\lambda_i}{w_i} > \alpha_o \qquad \text{for } 1 \le i \le \ell \tag{3.10}$$

The following theorem now provides a saddle-point solution for vector case (i):

<u>Theorem 3.1</u>: For the communication problem of vector case (i) depicted in Figure 1.4, a saddle-point solution $(\gamma^*, \delta^*, \mu^*)$ exists over the space $\Omega_s \times D \times \vartheta$, where

i) the transmitter-receiver pair is

$$(\gamma^*, \delta^*) = \begin{cases} (A_o^* u \ , \ K_o^* z) & \text{w.p. } \frac{1}{2h^*} \\ \quad \vdots & \quad \vdots \\ (A_{h^*-1}^* u \ , \ K_{h^*-1}^* z) & \text{w.p. } \frac{1}{2h^*} \\ (-A_o^* u \ , \ -K_o^* z) & \text{w.p. } \frac{1}{2h^*} \\ \quad \vdots & \quad \vdots \\ (-A_{h^*-1}^* u \ , \ -K_{h^*-1}^* z) & \text{w.p. } \frac{1}{2h^*} \end{cases} \tag{3.11}$$

where $A_i^* = \tau H_i G U$, and $K_i^* = \Sigma A_i^{*\prime} (A_i^* \Sigma A_i^{*\prime} + W)^{-1}$

ii) the jamming policy is

$$v^* = \eta \tag{3.12}$$

where η is a zero mean Gaussian noise with covariance Λ^* such that $\Lambda^* = \pi \operatorname{diag}(\sigma - r_1, \ldots, \sigma - r_{h^*}, 0, \ldots, 0)\, \pi'$, and η is independent of all other random vectors.

iii) the saddle-point value of J is

$$J^* = \frac{\left(\sum_{i=1}^{\ell} \sqrt{\lambda_i w_i}\right)^2}{P + \sum_{i=1}^{\ell} w_i} + \sum_{i=\ell+1}^{n} \lambda_i \tag{3.13}$$

<u>Proof</u>: Let us first rewrite the saddle-point inequality,

$$J(\gamma^*, \delta^*, \mu) \leq J(\gamma^*, \delta^*, \mu^*) \leq J(\gamma, \delta, \mu^*) \;;\; \forall \gamma \in \Omega_s,\ \delta \in D,\ \mu \in \vartheta \tag{3.14}$$

In the following, we first verify the right-hand side inequality of (3.14) and then the left-hand side inequality.

<u>i) Right-Hand Side Inequality</u>: Given v^* as specified in (3.12), the sum of jamming vector and channel noise, r, becomes a zero mean Gaussian noise with covariance W. This then entails optimal transmission of an input vector reliably over the Gaussian vector channel to the receiver. So

$$z = x + \eta + r \tag{3.15}$$

where $\eta + r$ is additive zero-mean Gaussian noise with covariance W. This type of problem has been solved in [5] and [6]. Referring to [6], the optimal linear encoding policy is

$$x = A^{*}u \tag{3.16a}$$

where $A^{*} = TGU$ and $T'WT = \mathrm{diag}(\sigma,\ldots,\sigma,r_{h^*+1},\ldots,r_m)$ with $T'T = I$. A^{*} is not unique since the unitary transformation T is not unique; it could be any one of the matrices

$$\pi H_o, \pi H_1, \ldots, \pi H_{h^*-1} \tag{3.16b}$$

Let us verify the experssions for T given above. Firstly, for $i=0$, since $H_o = I$,

$$\begin{aligned}
&\pi H_o\ \mathrm{diag}(\sigma,\ldots,\sigma,r_{h^*+1},\ldots,r_m)H_o'\pi' \\
&= \pi\mathrm{diag}(\sigma,\ldots,\sigma,r_{h^*+1},\ldots r_m)\pi' \\
&= W
\end{aligned} \tag{3.17}$$

Secondly, for $i\neq 0$,

$$\begin{aligned}
&\pi H_i\ \mathrm{diag}(\sigma,\ldots,\sigma,r_{h^*+1},\ldots,r_m)H_i'\pi' \\
&= \pi \begin{pmatrix} T_o^i & 0 \\ 0 & I \end{pmatrix} \mathrm{diag}(\sigma,\ldots,\sigma,r_{h^*+1},\ldots,r_m) \begin{pmatrix} T_o^{i\prime} & 0 \\ 0 & I \end{pmatrix} \pi' \\
&= \pi \begin{pmatrix} T_o^i\sigma I T_o^{i\prime} & 0 \\ 0 & \mathrm{diag}(r_{h^*+1},\ldots,r_m) \end{pmatrix} \pi' \\
&= \pi\ \mathrm{diag}(\sigma,\ldots,\sigma,r_{h^*+1},\ldots,r_m)\pi' = W
\end{aligned} \tag{3.18}$$

Corresponding to each optimal linear encoding rule $x = A_i^* u$, the optimal estimator is linear:

$$\delta_i^*(z) = K_i^* z = \Sigma A_i^{*\prime}(A_i^* \Sigma A_i^{*\prime} + W)^{-1} z \tag{3.19}$$

The optimal value of J in (3.13) has been derived in [5] or [6].

ii) Left-Hand Side Inequality:

Given (γ^*, δ^*) as specified in (3.11), referring to [4] and conditioning on the particular structure of the transmitter and receiver chosen to be implemented, $\gamma_i(u) = A_i^* u$, and $\delta_i(z) = K_i^* z$, we obtain

$$\begin{aligned} J(\gamma,\delta,\mu \mid \gamma = A_i^* u, \delta = K_i^* z) = {} & E_\mu\{v' K_i^{*\prime} K_i^* v - 2v' K_i^{*\prime}(I - K_i^* A_i^*) E(u \mid v)\} \\ & + Tr\{(I - K_i^* A_i^*)\Sigma(I - K_i^* A_i^*)' + K_i^* R K_i^{*\prime}\} \end{aligned} \tag{3.20}$$

Then, unconditioning (3.20) under the probability distribution specified in (3.11), the second term in the first expectation of (3.20) cancels out (since both pairs (A_i^*, K_i^*) and $(-A_i^*, -K_i^*)$ appear with equal probability), leaving

$$J(\gamma,\delta,\mu) = \sum_{i=0}^{h^*-1} \frac{1}{h^*} E_\mu\{v' K_i^{*\prime} K_i^* v\} + \sum_{i=0}^{h^*-1} \frac{1}{h^*} Tr\{I - K_i^* A_i^*)\Sigma(I - K_i^* A_i^*)' + K_i^* R K_i^{*\prime}\} \tag{3.21}$$

Since only the first term in (3.21) depends on the structure of the jamming policy, the maximization problem is reduced to the following equivalent maximization problem:

$$\max_{\mu \in \vartheta} \sum_{i=0}^{h^*-1} E\,[v' K_i^{*\prime} K_i^* v] \quad , \quad \text{s.t. } Tr\{\Lambda\} \le Q \tag{3.22}$$

Let us first write out $K_i^{*\prime} K_i^*$:

$$K_i^{*\prime} K_i^* = \pi H_i \ \mathrm{diag}\left(\frac{\alpha_o}{w_1}\left(\sqrt{\frac{\lambda_1 w_1}{\alpha_o}} - w_1\right) \ \ldots \ \frac{\alpha_o}{w_\ell}\left(\sqrt{\frac{\lambda_\ell w_\ell}{\alpha_o}} - w_\ell\right) \ 0 \ \ldots \ 0\right) H_i^\prime \pi^\prime \qquad (3.23)$$

<u>Fact I</u>: $\frac{\alpha_o}{w_i}\left(\sqrt{\frac{\lambda_i w_i}{\alpha_o}} - w_i\right) \geq \frac{\alpha_o}{w_j}\left(\sqrt{\frac{\lambda_j w_j}{\alpha_o}} - w_j\right)$ for $i < j$

Now, to investigate how H_i affects the structure of the diagonal matrix in (3.23),

let $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, and note that since $H_i = \begin{pmatrix} T_o^i & 0 \\ 0 & I \end{pmatrix}$,

we have $H_i B H_i^\prime = \begin{pmatrix} T_o^i B_{11} T_o^{i\prime} & 0 \\ 0 & B_{12} \end{pmatrix}$

Taking B_{11} diagonal, we obtain

$$T_o B_{11} T_o^\prime = \mathrm{diag}\ (b_{h^*,h^*} \ , \ b_{1,1}, \ldots, b_{h^*-1,h^*-1})$$

<u>Fact II</u>: $T_o^i B_{11} T_o^{i\prime} = \mathrm{diag}(b_{h^*+1-i,h^*+1-i}, \ldots, b_{1,1}, \ldots, b_{h^*-i,h^*-i})$

Also, using the transformation $\eta = \pi^\prime v$ in (3.22), we obtain

$$\max_{u \in \vartheta} \sum_{i=0}^{h^*-1} E\left(\eta^\prime H_i \ \mathrm{diag}\ \left(\frac{\alpha_o}{w_1}\left(\sqrt{\frac{\lambda_1 w_1}{\alpha_o}} - w_1\right) \ \ldots \ \frac{\alpha_o}{w_\ell}\left(\sqrt{\frac{\lambda_\ell w_\ell}{\alpha_o}} - w\ \right) \ 0 \ \ldots \ 0\right) H_i^\prime \eta\right) \qquad (3.24)$$

subject to $\mathrm{Tr}\{\Lambda\} \leq Q$

where $\eta = (\eta_1 \ \ldots \ \eta_m)^\prime$

Now, we have two cases which need to be considered separately here:

a) $h^* < \ell$ and b) $h^* \geq \ell$

a) For $h^* < \ell$, using Fact II in (3.24), we obtain

$$\max_{\mu \in \vartheta} \left\{ \left[\sum_{j=1}^{h^*} \frac{\alpha_o}{w_j} \left(\sqrt{\frac{\lambda_i w_i}{\alpha_o}} - w_j \right) \right] \sum_{i=1}^{h^*} E(\eta_i^2) + \sum_{i=h^*-1}^{\ell} \frac{\alpha_o}{w_i} \left(\sqrt{\frac{\lambda_i w_i}{\alpha_o}} - w_i \right) E(\eta_i^2) \right\} \tag{3.25}$$

subject to $Tr\{N\} \leq Q$ where $N = E(\eta\eta')$.

Using Fact I, the optimal solution will satisfy

$$\sum_{i=1}^{h^*} E(\eta_1^{*2}) = Q$$

One of several possible optimal solutions will be:

$$E(\eta_i^{*2}) = \sigma - r_i \qquad \text{for } 1 \leq i \leq h^*$$

and $$E(\eta_i \eta_j) = 0 \qquad \text{for } i \neq j, \text{ or } i > h^*$$

In conclusion, $N^* = \mathrm{diag}(\sigma - r_1, \ldots, \sigma - r_{h^*}, 0, \ldots, 0)$

so $\Lambda^* = \pi N^* \pi$, $v^* = \pi \eta^*$

hence, v^* is a zero mean Gaussian noise with covariance Λ^*.

b) For $h^* \geq \ell$, (3.24) becomes

$$\max_{\mu \in \vartheta} \left[\sum_{j=1}^{\ell} \frac{\alpha_o}{w_j} \left(\sqrt{\frac{\lambda_j w_j}{\alpha_o}} - w_j \right) \right] \sum_{i=1}^{h^*} E(\eta_i^2) \qquad \text{s.t. } Tr\{N\} \leq Q \tag{3.26}$$

where η_i and N admit the same definition as in a).

The only difference here is that the second term in (3.25) is absent; however, the maximization problem does not depend on that term. So the same conclusion can be drawn as in a). This then completes verification of the optimal jamming policy in (3.12). □

We note at this point that expressions given in (3.16a) and (3.16b) do not cover all choices of the optimal encoding policy, given that v^* in (3.12) is adopted by the jammer. By the property of interchangability of the saddle-point solutions [2], other possible saddle-point policies for the transmitter-receiver pair will have essentially the same form as (3.11), so that any of these policies will lead to an objective function similar to (3.25) or (3.26), in which each element of $\{E(n_i^2),\ i=1,\ldots h^*\}$ has the same coefficient.

The following corollary now gives a set of necessary conditions for a saddle-point policy for the jammer. We present it without any proof since it is similar to that of the scalar case, with (3.20).

Corollary 3.1: If $\tilde{\mu}$ is a saddle-point policy for the jammer, then the jamming variable, v, generated by the probability distribution, $\tilde{\mu}$ will satisfy the following equality, for $i=0,\ldots,h^*-1$.

$$E\{v' K_i^{*\prime}(I - K_i^* A_i^*)u\} = 0 \tag{3.27}$$

$\square$

3.2. Numerical Examples on Vector Case (i)

In this section, we present some numerical examples to illustrate Theorem 3.1.

Example 1: Here we take the dimension of the input vector to be two, n=2 and the number of channels to be three, m=3. Let the covariance of input vector be

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

and the channel noise covariance be

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Further, set the power constraint of the encoder to be unity, P=1, and the power constraint of the jammer to be two, Q=2. Using Theorem 3.1, we obtain the saddle-point solution as follows:

$$\text{i)} \quad (\gamma^*, \delta^*) = \begin{cases} \left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u, \begin{pmatrix} \frac{\sqrt{2}}{3.5} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z \right) & \text{w.p. } 1/4 \\ \left(\begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u, \begin{pmatrix} \frac{-\sqrt{2}}{3.5} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z \right) & \text{w.p. } 1/4 \\ \left(\begin{pmatrix} 0 & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} u, \begin{pmatrix} 0 & \sqrt{2}/3.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} z \right) & \text{w.p. } 1/4 \\ \left(\begin{pmatrix} 0 & 0 \\ -1/\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} u, \begin{pmatrix} 0 & -\sqrt{2}/3.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} z \right) & \text{w.p. } 1/4 \end{cases}$$

ii) v^* is a zero mean Gaussian noise with covariance Λ^*, where

$$\Lambda^* = \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

iii) the saddle-point value of J is $J^* \simeq 2.43$.

In this example, the optimal policy dictates the transmitter to send only the first element of the input vector but through either one of first two channels such that each one of them will be used with equal probability, since the jammer has equalized the first two eigenvalues of

channel noises. If the power of the jammer is not large enough to equalize more than one eigenvalue of the channel noise, then this example can be reduced to the scalar problem (which has been treated in Section 2.2).

On the other hand, the mixing policy of the transmitter and receiver pair in this problem will admit the equilization jamming policy in equilibrium with itself. Since the jammer cannot predict which channel has transmitted the signal, he can only do his best by trying to equalize over all the eigenvalues of the channel noise.

Example 2: We take n, m, z and R as in Example 1, but with P=2 and Q=3. Again, using Theorem 3.1, we obtain

$$
\text{i)}\quad (\gamma^*,\delta^*) = \begin{cases}
\left(\begin{pmatrix} \left(\dfrac{5\sqrt{2}-3}{2(1+\sqrt{2})}\right)^{1/2} & 0 \\ 0 & \left(\dfrac{5-3\sqrt{2}}{1+\sqrt{2}}\right)^{1/2} \\ 0 & 0 \end{pmatrix} u, \right. \\
\left. \begin{pmatrix} \dfrac{((5\sqrt{2}-3)(1+\sqrt{2}))^{1/2}}{8} & 0 & 0 \\ 0 & \dfrac{((5-3\sqrt{2})(1+\sqrt{2}))^{1/2}}{8} & 0 \end{pmatrix} z \right) & \text{w.p. } 1/6 \\
\left(\begin{pmatrix} -\left(\dfrac{5\sqrt{2}-3}{2(1+\sqrt{2})}\right)^{1/2} & 0 \\ 0 & -\left(\dfrac{5-3\sqrt{2}}{1+\sqrt{2}}\right)^{1/2} \\ 0 & 0 \end{pmatrix} u, \right. \\
\left. \begin{pmatrix} -\dfrac{((5\sqrt{2}-3)(1+\sqrt{2}))^{1/2}}{8} & 0 & 0 \\ 0 & -\dfrac{((5-3\sqrt{2})(1+\sqrt{2}))^{1/2}}{8} & 0 \end{pmatrix} z \right) & \text{w.p. } 1/6
\end{cases}
$$

$$
\begin{cases}
\left(\begin{pmatrix} 0 & 0 \\ \left(\frac{5\sqrt{2} - 3}{2(1 + \sqrt{2})}\right)^{1/2} & 0 \\ 0 & \left(\frac{5 - 3\sqrt{2}}{1 + \sqrt{2}}\right)^{1/2} \end{pmatrix} u, \right. \\
\left. \begin{pmatrix} 0 & \frac{((5\sqrt{2} - 3)(1 + \sqrt{2}))^{1/2}}{8} & 0 \\ 0 & 0 & \frac{((5 - 3\sqrt{2})(1 + \sqrt{2}))^{1/2}}{8} \end{pmatrix} z \right) & \text{w.p. } 1/6 \\
\left(\begin{pmatrix} 0 & 0 \\ -\left(\frac{5\sqrt{2} - 3}{2(1 + \sqrt{2})}\right)^{1/2} & 0 \\ 0 & -\left(\frac{5 - 3\sqrt{2}}{1 + \sqrt{2}}\right)^{1/2} \end{pmatrix} u, \right. \\
\left. \begin{pmatrix} 0 & -\frac{((5\sqrt{2} - 3)(1 + \sqrt{2}))^{1/2}}{8} & 0 \\ 0 & 0 & -\frac{((5\sqrt{2} - 3)(1 + \sqrt{2}))^{1/2}}{8} \end{pmatrix} z \right) & \text{w.p. } 1/6 \\
\left(\begin{pmatrix} 0 & \left(\frac{5 - 3\sqrt{2}}{1 + \sqrt{2}}\right)^{1/2} \\ 0 & 0 \\ \left(\frac{5\sqrt{2} - 3}{2(1 + \sqrt{2})}\right)^{1/2} & 0 \end{pmatrix} u, \right. \\
\left. \begin{pmatrix} 0 & 0 & \frac{((5\sqrt{2} - 3)(1 + \sqrt{2}))^{1/2}}{8} \\ \frac{((5 - 3\sqrt{2})(1 + \sqrt{2}))^{1/2}}{8} & 0 & 0 \end{pmatrix} z \right) & \text{w.p. } 1/6
\end{cases}
$$

$$
\left\{ \begin{array}{l}
\left(\begin{pmatrix} 0 & -\left(\frac{5 - 3\sqrt{2}}{1 + \sqrt{2}}\right)^{1/2} \\ 0 & 0 \\ -\left(\frac{5\sqrt{2} - 3}{2(1 + \sqrt{2})}\right)^{1/2} & 0 \end{pmatrix} u \right. \\
\left. \begin{pmatrix} 0 & 0 & -\frac{((5\sqrt{2} - 3)(1 + \sqrt{2}))^{1/2}}{8} \\ -\frac{((5 - 3\sqrt{2})(1 + \sqrt{2}))^{1/2}}{8} & 0 & 0 \end{pmatrix} z \right) \\
\qquad \qquad \qquad \text{w.p. } 1/6
\end{array} \right.
$$

ii) v^* is a zero mean Gaussian noise with covariance Λ^*, where

$$
\Lambda^* = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

iii) the saddle-point value of J is $J^* \cong 2.1857$.

Again, the optimal policy dictates the transmitter to use two of the three channels simultaneously at any time such that each channel has the same probability to be accessed by each of two elements of the input vector. Hence, it again admits the equilization jamming policy in equilibrium with (γ^*, δ^*).

Example 3: Set n=3, m=2

$$
\Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}
$$

$$
P=1 \quad , \quad Q=7
$$

The saddle-point solution becomes (again using Theorem 3.1)

$$\text{i)}\quad (\gamma^*,\delta^*) = \begin{cases} \left(\begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u, \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} z \right) & \text{w.p. } 1/4 \\ \left(\begin{pmatrix} -1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u, \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} z \right) & \text{w.p. } 1/4 \\ \left(\begin{pmatrix} 0 & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \end{pmatrix} u, \begin{pmatrix} 0 & \frac{1}{2\sqrt{3}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} z \right) & \text{w.p. } 1/4 \\ \left(\begin{pmatrix} 0 & 0 & 0 \\ -1/\sqrt{3} & 0 & 0 \end{pmatrix} u, \begin{pmatrix} 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} z \right) & \text{w.p. } 1/4 \end{cases}$$

ii) v^* *is zero mean Gaussian vector with covariance* $\Lambda^* = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$

iii) the saddle-point value is $J^* = 4\frac{1}{3}$.

The result here is in the same spirit as that of Example 1.

<u>Example 4</u>: Take m, n, Σ, R and P as in Example 3, but with Q=0.5.

Working with Theorem 3.1, we obtain:

$$\text{i)}\quad (\gamma^*,\sigma^*) = \begin{cases} (Gu,\ Kz) & \text{w.p. } 1/2 \\ (-Gu,\ -Kz) & \text{w.p. } 1/2 \end{cases}$$

$$\text{where } G = \begin{pmatrix} \left(\frac{\sqrt{4.5}-1}{\sqrt{4.5}+2}\right)^{1/2} & 0 & 0 \\ 0 & \left(\frac{5-2\sqrt{4.5}}{4+2\sqrt{4.5}}\right)^{1/2} & 0 \end{pmatrix}$$

and

$$K = \begin{pmatrix} \dfrac{((\sqrt{4.5} - 1)(2 + \sqrt{4.5}))^{1/2}}{1.5\sqrt{4.5}} & 0 \\ 0 & \dfrac{\sqrt{2}}{5}((5 - 2\sqrt{4.5})(2 + \sqrt{4.5}))^{1/2} \\ 0 & 0 \end{pmatrix}$$

ii) v^* is a zero mean Gaussian noise with covariance

$$\Lambda^* = \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix}$$

iii) $J^* \simeq 4.7745.$

In this last example, the element of the input vector corresponding to the smallest eigenvalue has been ignored totally. Since the power of the jammer is not enough to equalize the two eigenvalues of the channel noise, he allocates all his power to the channel corresponding to the smallest eigenvalue. The overall optimization problem for the transmitter and receiver pair will determine globally how to distribute the total energy over the first two elements of the input vector. Then each element with his own assigned amount of energy plays his own game locally. Hence, the result can be obtained by solving two independent scalar problems in parallel, under a specified power allocation between the two channels.

3.3. Derivation of Max.Min. Solution in Vector Case (ii)

We now direct our attention to vector case (ii), and discuss the derivation of its max.min solution. We do this by first deriving an upper bound on the max.min. value of the original problem by restricting the encoding and decoding policies to a smaller domain, and then verifying that

for some jamming policy, this upper bound is actually attainable for the original problem.

Let us first introduce here a new matrix π_R. π_R is an elementary matrix, ([7]), which simply reorders the diagonal elements of $\pi'(\Lambda + R)\pi$ (π has been defined in (3.1)) in increasing order (but not necessarily diagonalizes $\pi'(\Lambda + R)\pi$), i.e.,

$$\pi_R' \pi'(\Lambda + R)\pi\pi_R = \begin{pmatrix} e_1 & - & \cdots & - \\ - & e_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ - & \cdots & \cdots & e_m \end{pmatrix}; \quad 0 < e_1 \cdots \leq e_m \tag{3.28}$$

and

$$\pi_R' \pi_R = I$$

To clarify this point, let us illustrate the idea of π_R in the following example:

We wish to reorder the entries of $M = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$ in the new form $\begin{pmatrix} 3 & - & - \\ - & 7 & - \\ - & - & 5 \end{pmatrix}$.

The matrix π_R that does the job is $\pi_R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, whereby:

$$\pi_R' M \pi_R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 2 \\ 9 & 7 & 8 \\ 6 & 4 & 5 \end{pmatrix}$$

and $\pi_R' \pi_R$,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By following through the lines of the above calculation, we claim that an appropriate π_R always exists. For convenience, let Π stand for the set of those elementary matrices. Note that if π_R is an element of Π, then its negative, $-\pi_R$ also belongs to Π.

Now, coming back to our problem, fix jammer's policy at $\mu \in \vartheta$, and choose

$$\gamma(u) = Au \quad , \quad \delta(z) = Kz \tag{3.29}$$

where

$$A = \pi\pi_R GU \quad , \quad K = \Sigma A'(A \Sigma A' + W)^{-1} \tag{3.30}$$

(U was defined in 3.2.)

with

$$\pi_R'\pi' W\pi\pi_R = \mathrm{diag}(\sigma, \ldots \sigma, r_{h^*+1}, \ldots, r_m) = \mathrm{diag}\ (w_1 \ldots w_m) \tag{3.31}$$

The remaining notation used in the sequel is the same as that introduced earlier at the beginning of Section 3.1. Now the only freedom left in the minimization of J over the restricted domain of encoding and decoding policies is to choose an optimal π_R from the set, Π. Algebraically,

$$\max_{\mu \in \vartheta}.\ \min_{\substack{\gamma \in \Omega_s \\ \delta \in D}}.\ J \leq \max_{\mu \in \vartheta}.\ \min_{\pi_R \in \Pi}.\ J \tag{3.32}$$

where $\qquad J = E\|\delta(\gamma(u) + v + r) - u\|^2$

Considering the right-hand side of (3.32),

$$\max_{\mu \in \vartheta}.\ \min_{\pi_R \in \Pi}.\ J = \max_{\mu \in \vartheta}.\ \min_{\pi_R \in \Pi}.\ E\{v'K'Kv - 2v'K'(I - KA)u\} + \mathrm{Tr}\{(I - KA)\Sigma(I - KA)' + KRK'\} \tag{3.33}$$

Note that $KA = \Sigma A'(A \Sigma A' + W)^{-1}A$

$$= \Sigma U'G'\pi_R'\pi'(\pi\pi_R GU \Sigma U'G'\pi_R'\pi' + W)^{-1}\pi\pi_R GU \tag{3.34}$$

$$KA = \Sigma U'G'(GU \Sigma U'G' + \pi_R'\pi'W\pi\pi_R)^{-1}GU$$

The above expression (3.34) is constant with respect to either the maximization or minimization of J in (3.33) since we had set $\pi_R'\pi'W\pi\pi_R$ to be a constant matrix in (3.31). Here, let $E(uv') = N$; then the max.min. problem in (3.33) is reduced to the following equivalent problem:

$$\max_{\substack{N,\Lambda \\ Tr\{\Lambda\} \le Q}} \min_{\pi_R \in \Pi} \tilde{J} = \max_{\substack{N,\Lambda \\ Tr\{\Lambda\} \le Q}} \min_{\pi_R \in \Pi} Tr\{\Lambda K'K - 2K'(I - KA)N + RK'K\} \tag{3.35}$$

$$= \max_{\substack{N,\Lambda \\ Tr\{\Lambda\} \le Q}} \min_{\pi_R \in \Pi} [Tr\{(\Lambda + R)K'K\} - 2Tr\{K'(I - KA)N\}]$$

Since the second trace in (3.35) can be always be made positive by an appropriate choice of π_R, and also setting N=0 does not affect the choice of Λ, the jammer will set N=0 in (3.35). Hence, the problem further reduces to

$$\max_{\mu \in \vartheta} \min_{\pi_R \in \Pi} \tilde{\tilde{J}} = \max_{\mu \in \vartheta} \min_{\pi_R \in \Pi} Tr\{(\Lambda + R)K'K\} \tag{3.36}$$

Now, writing out $K'K$, we have

$$K'K = \pi\pi_R \operatorname{diag}\left(\frac{\alpha_o}{w_1}\left(\sqrt{\frac{\lambda_1 w_1}{\alpha_o}} - w_1\right) \ldots \frac{\alpha_o}{w_\ell}\left(\sqrt{\frac{\lambda_\ell w_\ell}{\alpha_o}} - w_\ell\right)0\ldots0\right)\pi_R'\pi' \tag{3.37}$$

Recall Fact I in Theorem 3.1

$$\frac{\alpha_o}{w_i}\left(\sqrt{\frac{\lambda_i w_i}{\alpha_o}} - w_i\right) \geq \frac{\alpha_o}{w_j}\left(\sqrt{\frac{\lambda_j w_i}{\alpha_o}} - w_j\right) \qquad \text{for } i<j$$

in view of which the diagonal elements in (3.37) are in decreasing order. Rewriting J in (3.36), we have

$$\begin{aligned}
\underset{\substack{\Lambda \\ Tr\{\Lambda\}\leq Q}}{\max.}\ \underset{\pi_R \in \Pi}{\min.}\ \tilde{\tilde{J}} &= \underset{\substack{\Lambda \\ Tr\{\Lambda\}\leq Q}}{\max.}\ \underset{\pi_R \in \Pi}{\min.}\ Tr\{\pi_R'\pi'(\Lambda + R)\pi\pi_R \operatorname{diag}\left(\frac{\alpha_o}{w_1}\left(\sqrt{\frac{\lambda_1 w_1}{\alpha_o}} - w_1\right) \ldots \right. \\
&\qquad \left. \frac{\alpha_o}{w_\ell}\left(\sqrt{\frac{\lambda_\ell w_\ell}{\alpha_o}} - w_\ell\right)0\ldots 0\right)\} \\
&= \underset{\substack{\Lambda \\ Tr\{\Lambda\}\leq Q}}{\max.}\ \underset{\pi_R \in \Pi}{\min.}\ Tr\{\pi_R'(\pi'\Lambda\pi + \Pi_R'\Pi)\pi_R \operatorname{diag}\left(\frac{\alpha_o}{w_1}\left(\sqrt{\frac{\lambda_1 w_1}{\alpha_o}} - w_1\right) \ldots \right. \\
&\qquad \left. \frac{\alpha_o}{w_\ell}\left(\sqrt{\frac{\lambda_\ell w_\ell}{\alpha_o}} - w\ \right)0\ldots 0\right)\} \\
&= \underset{\substack{\Lambda \\ Tr\{\Lambda\}\leq Q}}{\max.}\ \underset{\pi_R \in \Pi}{\min.}\ Tr\{\pi_R'(\pi'\Lambda\pi + \operatorname{diag}(r_1,\ldots,r_m))\pi_R \\
&\qquad \operatorname{diag}\left(\frac{\alpha_o}{w_1}\left(\sqrt{\frac{\lambda_1 w_1}{\alpha_o}} - w_1\right) \ldots \frac{\alpha_o}{w_\ell}\left(\sqrt{\frac{\lambda_\ell w_\ell}{\alpha_o}} - w_\ell\right)0\ldots 0\right)\}
\end{aligned} \tag{3.38}$$

Obviously, from the above expression in (3.38) and the property of π_R (which always orders the diagonal elements of $\pi'\Lambda\pi + \operatorname{diag}(r_1 \ldots r_m)$ in increasing order), in addition to the Fact I, the best jammer can do is to equalize the eigenvalues of the diagonal matrix $\operatorname{diag}(r_1 \ldots r_m)$. Hence, the optimal covariance matrix Λ^* satisfies

$$\pi'\Lambda^*\pi = \begin{pmatrix} \sigma-r_1 & - & - & \cdots & - \\ - & \ddots & & & \vdots \\ \vdots & & \sigma-r_{h^*} & & \vdots \\ \vdots & & & 0 & \vdots \\ \vdots & & & & \ddots \\ - & \cdots & \cdots & \cdots & 0 \end{pmatrix} \quad (3.39)$$

Using (3.39) in (3.12), it can be shown by mimicking the arguments used in the proof of the L.H.S. inequality of Theorem 3.1, that the upper bound in (3.32) (max.min. value in the restricted problem) is the same as the saddle-point value in Theorem 3.1. Then, (3.32) becomes

$$\max_{\mu\in\vartheta}\ \min_{\substack{\gamma\in\Omega_s \\ \delta\in D}} J \leq \text{saddle-point value of vector case (i)} \quad (3.40)$$

Now, the next step is to verify the attainability of the upper bound in (3.40) at some jamming policy. In a straightforward fashion, by choosing v^* the same as in (3.12) of Theorem 3.1, the minimization of J over the space $\Omega_s xD$ will be the same as in the proof of the right-hand side of the saddle-point inequality in Theorem 3.1. Hence, the upper bound in (3.40) is achievable for at least one jamming policy. We summarize the above results in the following theorem, which uses the notation introduced at the beginning of Section 3.1.

Theorem 3.2: For the communication system of vector case (ii), there exist two max.min. solutions $(\gamma^*,\delta^*,\beta^*)$ and $(-\gamma^*,-\delta^*,-\beta^*)$ where

i) $\gamma^*(u) = A^*u = \pi GUu$ (3.41)

ii) $v^* = \eta$ (3.42)

where η is a zero mean Gaussian noise with covariance Λ^* such that $\Lambda^* = \Pi\,\text{diag}(\sigma-r_1,\ldots,\sigma-r_{h^*},0\ldots0)\pi'$, and η is independent of all other random vectors.

iii) $\delta^*(z) = \Sigma A^{*\prime}(A^* \Sigma A^{*\prime} + W)^{-1} z = K^* z$ (3.43)

iv) $$\underline{J}^* = \frac{\left(\sum_{i=1}^{\ell} \sqrt{\lambda_i w_i}\right)^2 + \sum_{i=\ell+1}^{n} \lambda_i}{P + \sum_{i=1}^{\ell} w_i} \qquad (3.44)$$

□

Here, π,U,G,W,σ, and h^* are as defined by (3.1), (3.2), (3.8), (3.5), (3.6), and (3.7), respectively.

The max.min. solution presented in Theorem 3.2 is in almost the same form as the saddle-point solution of vector case (i) (in Theorem 3.1). The only difference is that (γ^*,δ^*) in (3.41) and (3.43) is one of the realizations of (γ^*,δ^*) in (3.11).

3.4. Derivation of a Min.Max. Solution for Vector Case (ii)

In this section, we present a min.max. solution in closed explicit form in some parametric regions. (But unfortunately, for certain parametric regions, we still do not have a closed explicit forms for a min.max. solution, and at this point this seems to be quite impossible.)

Before presenting the main result in Theorem 3.3, we introduce some new notation (in addition to those introduced earlier):

Define

$$\sqrt{\alpha_1} = \frac{\sum_{i=1}^{\tilde{\ell}} \sqrt{\lambda_i r_i}}{(\sqrt{P} - \sqrt{Q})^2 + \sum_{i=1}^{\tilde{\ell}} r_i} \qquad (3.45)$$

where $\tilde{\ell}$ is the maximum integer which satisfies

$$\frac{\lambda_i}{r_i} > \alpha_1 \qquad \text{for } 1 \le i \le \tilde{\ell} \tag{3.46}$$

Furthermore, let $\tilde{G} = [\tilde{g}_{ij}]$

$$\tilde{g}_{ii} = \begin{cases} \left(\sqrt{\dfrac{\lambda_i r_i}{\alpha_1}} - r_i\right)^{1/2} / \sqrt{\lambda_i} & \text{for } i \le \tilde{\ell} \\ 0 & \text{otherwise} \end{cases} \tag{3.47}$$

Let

$$\tilde{A}^* = \pi \tilde{G} U \tag{3.48}$$
$$\tilde{K}^* = \Sigma \tilde{A}^{*\prime} (\tilde{A}^* \Sigma \tilde{A}^{*\prime} + R)^{-1}$$

where π and U are defined as in (3.1) and (3.2), respectively. Finally, introduce the parametric regions, R_5, R_6,

$$\begin{cases} R_5 : P \le Q \\ R_6 : P > Q \text{ and } Q < \dfrac{\alpha_1 r_1}{\lambda_1} P \end{cases} \tag{3.49}$$

Theorem 3.3: For the communication system of vector case (ii), there exist two min.max. solutions, $(\gamma^*, \delta^*, \beta^*)$ and $(-\gamma^*, -\delta^*, -\beta^*)$, respectively, for the regions R_5 and R_6, over the space $\Omega_s x D x \vartheta$, where

i) $$\gamma^*(u) = \begin{cases} \text{arbitrary} & \text{in } R_5 \\ \dfrac{1}{1 - \sqrt{\dfrac{Q}{P}}} \tilde{A}^* u & \text{in } R_6 \end{cases} \tag{3.50}$$

ii) $$\varepsilon^*(u) = \begin{cases} \gamma^*(-u) & \text{in } R_5 \\ \dfrac{-\sqrt{\frac{Q}{P}}}{1-\sqrt{\frac{Q}{P}}}\tilde{A}^* u & \text{in } R_6 \end{cases} \tag{3.51}$$

iii) $$\delta^*(z) = \begin{cases} 0 & \text{in } R_5 \\ \tilde{K}^* z & \text{in } R_6 \end{cases} \tag{3.52}$$

iv) $$\bar{J}^* = \begin{cases} \sum_{i=1}^{n} \lambda_i & \text{in } R_5 \\ \dfrac{\left(\sum_{i=1}^{\tilde{\ell}} \sqrt{\lambda_i r_i}\right)^2}{(\sqrt{P}-\sqrt{Q})^2 + \sum_{i=1}^{\tilde{\ell}} r_i} + \sum_{i=\tilde{\ell}+1}^{n} \lambda_i & \text{in } R_6 \end{cases} \tag{3.53}$$

Proof: The proof of the min.max. solution in Region R_5 is just straightforward vector extension of the proof for the scalar problem in Theorem 2.3; hence, we go directly to the proof for the Region R_6.

Analogous to the case of Theorem 3.2, we first derive a lower bound for the original min.max. problem and then verify the attainability of the lower bound at some encoding and decoding policies for the original problem. First we set up the inequality,

$$\underset{\substack{\gamma \in \Omega \\ \delta \in D^s}}{\text{min.}} \ \underset{\mu \in \vartheta}{\text{max.}} \ J \geq \underset{\substack{\gamma \in \Omega \\ \delta \in D}}{\text{min.}} \ J\left(\gamma(u), \delta(z), v = -\frac{Q}{P} E(x|u)\right) \tag{3.54}$$

where x is the random variable generated by $\gamma(u)$.

Considering the right-hand side of (3.54), z becomes

$$z = x - \sqrt{\frac{Q}{P}}\, E(x|u) + r \tag{3.55}$$

Now, rewrite z,

$$z = (1 - \sqrt{\frac{Q}{P}})x + \sqrt{\frac{Q}{P}}(x - E(x|u)) + r \tag{3.56}$$

and consider a new minimization problem by splitting expression (3.56) into two parts: z_1 and z_2, where

$$\begin{aligned} z_1 &= (1 - \sqrt{\frac{Q}{P}})x + r \\ z_2 &= \sqrt{\frac{Q}{P}}(x - E(x|u)) \end{aligned} \tag{3.57}$$

Instead of z in (3.56), z_1 and z_2 in (3.57) will be received by the receiver. This new problem will achieve a minimum square error which is lower than the one of the problem defined by the right-hand side of (3.54). By manipulating on these data z_1 and z_2 algebraically, we obtain

$$\begin{cases} \hat{z}_1 \equiv \dfrac{z_1}{1 - \sqrt{\frac{Q}{P}}} = x + \dfrac{1}{1 - \sqrt{\frac{Q}{P}}}\, r \\ \hat{z}_2 \equiv \dfrac{z_2}{\sqrt{\frac{Q}{P}}} = x - E(x|u) \end{cases} \tag{3.58}$$

Further,

$$\begin{cases} \hat{z}_1 = x + \dfrac{1}{1 - \sqrt{\frac{Q}{P}}}\, r \\ \hat{z}_1 - \hat{z}_2 = E(x|u) + \dfrac{1}{1 - \sqrt{\frac{Q}{P}}}\, r \end{cases} \tag{3.59}$$

Since the problem in (3.57) has just been linearly transformed into the problem in (3.59), we claim that both minimization problems over the space $\Omega_s xD$ are equivalent. By inspection on (3.59), we see that the randomized encoding policy is totally unnecessary; hence, we can say that an optimal encoding policy exists as a deterministic function of u. Coming back to the problem in (3.57), $z_2 = 0$, and we are left with only $z_1 = (1 - \sqrt{\frac{Q}{P}})x + r$. This problem has been solved in [5] and [6], with the solution being $\gamma^*(u) = \tilde{A}^* u$, and $\delta^*(z) = \tilde{K}^* z$ with the mean square error, J^* specified in (3.53). To summarize, from (3.54), we have

$$\min_{\substack{\gamma \in \Omega_s \\ \delta \in D}} \ \max_{\mu \in \vartheta} \ J \geq \frac{\left(\sum_{i=1}^{\tilde{\ell}} \sqrt{\lambda_i r_i} \right)^2}{(\sqrt{P} - \sqrt{Q})^2 + \sum_{i=1}^{\tilde{\ell}} r_i} + \sum_{i=\tilde{\ell}+1}^{n} \lambda_i \tag{3.60}$$

Now to complete the proof, we have to verify the attainability of the lower bound in (3.57).

Starting with the encoding and decoding policies in (3.50) and (3.52), respectively, the maximization of J becomes

$$\max_{\mu \in \vartheta} \ J \left(\frac{1}{1 - \sqrt{\frac{Q}{P}}} \tilde{A}^* u \ , \ \tilde{K}^* z, \mu \right)$$

$$= \max_{\mu \in \vartheta} \ E\{v' \tilde{K}^{*\prime} \tilde{K}^* v - 2v' \tilde{K}^{*\prime} (I - \frac{1}{1 - \sqrt{\frac{Q}{P}}} \tilde{K}^* \tilde{A}^*) u\}$$

$$+ \operatorname{Tr} \left(I - \frac{\tilde{K}^* \tilde{A}^*}{1 - \sqrt{\frac{Q}{P}}}\right) \Sigma \left(I - \frac{\tilde{K}^* \tilde{A}^*}{1 - \sqrt{\frac{Q}{P}}}\right)' + \tilde{K}^* R \tilde{K}^{*\prime}\} \tag{3.61}$$

Since the trace in (3.62) does not depend on the distribution μ, the problem reduces to

$$\max_{\mu\in\vartheta} \tilde{J} = \max_{\mu\in\vartheta} E\{v'\tilde{K}^{*\prime}\tilde{K}^{*}v - 2v\tilde{K}^{*\prime}(I - \frac{\tilde{K}^{*}\tilde{A}^{*}}{1-\sqrt{\frac{Q}{P}}})u\} \tag{3.62}$$

Firstly, write out $\tilde{K}^{*\prime}\tilde{K}^{*}$ and $\tilde{K}^{*\prime}(I - \frac{\tilde{K}^{*}\tilde{A}^{*}}{1-\sqrt{\frac{Q}{P}}})$,

$$\tilde{K}^{*\prime}\tilde{K}^{*} = \pi\,\mathrm{diag}\left(\frac{\alpha_1}{r_1}\left(\sqrt{\frac{\lambda_1 r_1}{\alpha_1}} - r_1\right)\dots\frac{\alpha_1}{r_{\tilde{\ell}}}\left(\sqrt{\frac{\lambda_{\tilde{\ell}} r_{\tilde{\ell}}}{\alpha_1}} - r_{\tilde{\ell}}\right)0\dots0\right)\pi' \tag{3.63}$$

$$\tilde{K}^{*\prime}(I - \frac{\tilde{K}^{*}\tilde{A}^{*}}{1-\sqrt{\frac{Q}{P}}}) = \pi\,\mathrm{diag}\left(\frac{\left(\sqrt{\frac{\alpha_1\lambda_1}{r_1}} - \alpha_1\right)^{1/2}\left(-\sqrt{Q} + \sqrt{\frac{\alpha_1 r_1 P}{\lambda_1}}\right)}{\sqrt{P} - \sqrt{Q}} \dots \frac{\left(\sqrt{\frac{\alpha_1\lambda_{\tilde{\ell}}}{r_{\tilde{\ell}}}} - \alpha\right)^{1/2}\left(-\sqrt{Q} + \sqrt{\frac{\alpha_1 r_{\tilde{\ell}} P}{\lambda_{\tilde{\ell}}}}\right)}{\sqrt{P} - \sqrt{Q}}\;0\dots0\right)U \tag{3.64}$$

Secondly, perform the linear transformation, $\eta' = v'\pi$ or $\eta = \pi' v$. Then, the problem in (3.62) becomes equivalent to

$$\begin{aligned}\max_{\eta} \tilde{\tilde{J}} = E\Bigg\{&\eta'\mathrm{diag}\left(\frac{\alpha_1}{r_1}\left(\sqrt{\frac{\lambda_1 r_1}{\alpha_1}} - r_1\right)\dots\frac{\alpha_1}{r_{\tilde{\ell}}}\left(\sqrt{\frac{\lambda_{\tilde{\ell}} r_{\tilde{\ell}}}{\alpha_1}} - r_{\tilde{\ell}}\right)0\dots0\right)\eta \\ &- 2\eta'\,\mathrm{diag}\left(\frac{\left(\sqrt{\frac{\alpha_1\lambda_1}{r_1}} - \alpha_1\right)^{1/2}\left(-\sqrt{Q} + \sqrt{\frac{\alpha_1 r_1 P}{\lambda_1}}\right)}{\sqrt{P} - \sqrt{Q}} \dots \frac{\left(\sqrt{\frac{\alpha_1\lambda_{\tilde{\ell}}}{r_{\tilde{\ell}}}} - \alpha_1\right)^{1/2}\left(-\sqrt{Q} + \sqrt{\frac{\alpha_1 r_{\tilde{\ell}} P}{\lambda_{\tilde{\ell}}}}\right)}{\sqrt{P} - \sqrt{Q}}0\dots0\right)Uu\Bigg\}\end{aligned} \tag{3.65}$$

subject to $E(\eta'\eta) \le Q$.

Now, let $\alpha_i^2 = E(\eta_i^2)$, $\tilde{u} = Uu$. Then the problem can be expressed as

$$\max_{\eta_i} \tilde{\tilde{J}} = \sum_{i=1}^{\tilde{\ell}} \frac{\alpha_1}{r_i}\left(\sqrt{\frac{\lambda_i r_i}{\alpha_1}} - r_i\right)\sigma_i^2 - 2 \sum_{i=1}^{\tilde{\ell}} \frac{\left(\sqrt{\frac{\alpha_i \lambda_i}{r_i}} - \alpha_1\right)^{1/2}\left(-\sqrt{Q} + \sqrt{\frac{P\alpha_i r_i}{\lambda_i}}\right)}{\sqrt{P} - \sqrt{Q}} E(\tilde{u}_i \eta_i) \tag{3.66}$$

By Cauchy-Schwartz inequality, for a fixed set $(\sigma_1 \ldots \sigma_{\tilde{\ell}})$,

$$\max_{\sigma_i} |E(\tilde{u}_i \eta_i)| = \sigma_i \sqrt{\lambda_i} \tag{3.67}$$

Since in Region R_6, $Q < \frac{\alpha_1 r_1}{\lambda_1} P$, the problem in (3.66) further reduces to

$$\max_{\substack{\sigma_i s \\ 1 \leq i \leq \tilde{\ell} \\ \sigma_1^2 + \ldots + \sigma_{\tilde{\ell}}^2 \leq Q}} \tilde{\tilde{J}} = \max_{\substack{\sigma_i s \\ 1 \leq i \leq \tilde{\ell} \\ \sigma_1^2 + \ldots + \sigma_{\tilde{\ell}}^2 \leq Q}} \left\{ \sum_{i=1}^{\tilde{\ell}} \overbrace{\frac{\alpha_1}{r_1}\left(\sqrt{\frac{\lambda_i r_i}{\alpha_1}} - r_i\right)}^{\eta_i}\sigma_i^2 + 2 \sum_{i=1}^{\tilde{\ell}} \overbrace{\frac{\sqrt{\lambda_i}\left(\sqrt{\frac{\alpha_i \lambda_i}{r_i}} - \alpha_1\right)^{1/2}\left(-\sqrt{Q} + \sqrt{\frac{P\alpha_i r_i}{\lambda_i}}\right)}{\sqrt{P} - \sqrt{Q}}}^{\zeta_i} \sigma_i \right\} \tag{3.68}$$

The above maximization problem admits a unique solution on $(\sigma_1 \ldots \sigma_{\tilde{\ell}})$ [4], which is

$$\sigma_i^* = \frac{\zeta_i}{(\alpha - \eta_i)} \quad , \quad i = 1, \ldots, \tilde{\ell} \tag{3.69}$$

where

$$\alpha = \frac{\sqrt{P} - \sqrt{Q}}{\sqrt{Q}}\, \alpha_1 \tag{3.70}$$

Thus,

$$\sigma_i^* = \frac{\sqrt{Q}}{\sqrt{P} - \sqrt{Q}} \left(\sqrt{\frac{\lambda_i r_i}{\alpha_1}} - r_i\right)^{1/2}, \quad i=1,\ldots,\tilde{\ell} \tag{3.71}$$

In conclusion, given γ^* and δ^* in (3.50) and (3.52), respectively, the maximization of J over ϑ admits a unique optimal jamming policy specified by (3.67) and (3.71).

The next step is to show that the unique optimal jamming policy obtained above is actually $-\dfrac{\sqrt{\frac{Q}{P}}}{1 - \sqrt{\frac{Q}{P}}}\, \tilde{A}^* u$ (or $-\sqrt{\frac{Q}{P}}\, x$).

By (3.67), $\eta_i^* = -\dfrac{\sigma_i^*}{\sqrt{\lambda_i}}\, \tilde{u}_i$, $0 \le i \le \tilde{\ell}$. Hence,

$$\eta^* = -\,\mathrm{diag}\left(\frac{\sigma_1^*}{\sqrt{\lambda_1}} \ \cdots\ \frac{\sigma_{\tilde{\ell}}^*}{\sqrt{\lambda_{\tilde{\ell}}}}\ 0\ldots0\right) Uu$$

leading to

$$v^* = \pi\eta^* = -\pi\ \mathrm{diag}\left(\frac{\sigma_1^*}{\sqrt{\lambda_1}} \ \cdots\ \frac{\sigma_{\tilde{\ell}}^*}{\sqrt{\lambda_{\tilde{\ell}}}}\ 0\ldots0\right) Uu$$

$$= -\frac{\sqrt{Q}}{\sqrt{P} - \sqrt{Q}}\,\pi \left(\left(\sqrt{\frac{\lambda_1 r_1}{\alpha_1}} - r_1\right)^{1/2} \Big/ \sqrt{\lambda_1} \ \cdots\ \left(\sqrt{\frac{\lambda_{\tilde{\ell}} r_{\tilde{\ell}}}{\alpha_1}} - r_{\tilde{\ell}}\right)^{1/2} \Big/ \sqrt{\lambda_{\tilde{\ell}}}\ \ 0\ldots0 \right) Uu$$

$$= -\frac{\sqrt{Q}}{\sqrt{P} - \sqrt{Q}}\,\pi\tilde{G}Uu \ ; \quad v^* = -\frac{\sqrt{Q}}{\sqrt{P} - \sqrt{Q}}\,\tilde{A}^* u \tag{3.72}$$

By referring back to (3.57) and noting that $x = \frac{1}{1 - \sqrt{\frac{Q}{P}}} \tilde{A}^* u$, we establish the attainability of the lower bound.

The proof of Theorem 3.3 is thus completed. □

Toward the end of this section, we will illustrate the spirit of Theorem 3.3 with an example.

Example: Take P=40, Q=10, $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $R = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 12 \end{pmatrix}$. Then, working through Equations (3.45)-(3.49), we obtain, if $\tilde{\ell} = 2$,

$$\sqrt{\alpha_1} = \frac{\sum_{i=1}^{2} \sqrt{\lambda_i r_i}}{(\sqrt{P} - \sqrt{Q})^2 + \sum_{i=1}^{2} r_i} = \frac{\sqrt{10} + \sqrt{11}}{(\sqrt{40} - \sqrt{10})^2 + 21} \cong 0.2090$$

$$\rightarrow \alpha_1 \cong 0.04368$$

checking the validity of $\tilde{\ell}=2$ with (3.46), we obtain

$$\frac{\lambda_2}{r_2} = \frac{1}{11} \cong 0.09091 > \alpha_1$$

Hence, $\tilde{\ell}=2$ is valid.

Now, checking with (3.49), to see whether $Q < \frac{\alpha_1 r_1}{\lambda_1} P$, we obtain

$$\frac{\alpha_1 r_1}{\lambda_1} P \cong 0.04368 \times 10 + 40 \cong 17.4720$$

so, indeed, Q = 10 < 17.472.

After checking all the necessary conditions, we apply Theorem 3.3 to obtain a min.max. solution.

Using (3.47) to obtain

$$\tilde{g}_{11} \cong \left(\sqrt{\frac{10}{0.04368}} - 10\right)^{1/2} / \sqrt{1} \cong 2.26510$$

$$\tilde{g}_{22} \cong \left(\sqrt{\frac{11}{0.04368}} - 11\right)^{1/2} / \sqrt{1} \cong 2.20663$$

$$\tilde{A}^* = \tilde{G}^* \cong \begin{pmatrix} 2.26510 & 0 \\ 0 & 2.20663 \\ 0 & 0 \end{pmatrix}$$

By (3.48),

$$\begin{aligned} \tilde{K}^* &= \Sigma \, \tilde{A}^{*\prime} (\tilde{A}^* \, \Sigma \, \tilde{A}^{*\prime} + R)^{-1} \\ &= \tilde{A}^{*\prime} (\tilde{A}^* \tilde{A}^{*\prime} + R)^{-1} \\ &= \begin{pmatrix} 0.1497 & 0 & 0 \\ 0 & 0.13905 & 0 \end{pmatrix} \end{aligned}$$

also, $\sqrt{\frac{Q}{P}} = \frac{1}{2}$.

Applying (3.50)-(3.53), we have a min.max. solution as given below

i) $\gamma^*(u) = 2 \begin{pmatrix} 2.26510 & 0 \\ 0 & 2.20663 \\ 0 & 0 \end{pmatrix} u$

ii) $\beta^*(u) = - \begin{pmatrix} 2.26510 & 0 \\ 0 & 2.20663 \\ 0 & 0 \end{pmatrix} u$

iii) $\delta^*(z) = \begin{pmatrix} 0.1497 & 0 & 0 \\ 0 & 0.13905 & 0 \end{pmatrix} z$

iv) $\bar{J}^* = 1.35411$

In the preceding example, both channels have been used and the smallest signal eigenvalue has not transmitted, as dictated by the optimality of the solution.

Also, one interesting special structure of vector case (ii) is worth mentioning here. When the communication system consists of only one channel, the min.max. problem becomes equivalent to the scalar min.max. problem of Section 2.3 since only the largest signal eigenvalue is transmitted through the single channel [5],[6], i.e., $\tilde{u} \equiv Uu$ and $\tilde{u} \equiv (\tilde{u}_1 \ldots \tilde{u}_n)$ s.t. only $\tilde{u}_1$ will be encoded and transmitted.

3.5. Summary of the Solutions in All Vector Cases

Let us now collect the results presented in the previous sections and present them comparatively in Table 3.1. The highlights of this comparison are the following:

i) The saddle-point value of J in vector case (i) is the same as the max.min. value of J, $\underline{J}^*$ in vector case (ii). Notationally, max.min. of vector case (i) $\Longleftrightarrow$ max.min. of vector case (ii).

ii) In Region R_5, the min.max. value of J, $\bar{J}^*$, for vector case (ii) is strictly greater than the max.min. value of J, $\underline{J}^*$, in vector case (ii) and the saddle-point value of J in vector case (i).

TABLE 3.1. SUMMARY OF THE RESULTS IN ALL VECTOR CASES

Vector Case (i)	Vector Case (ii)	
Saddle-Point Solution $(\gamma^*,\delta^*,\beta^*)$	Max.Min. Solution $(\gamma^*,\delta^*,\beta^*)$ & $(-\gamma^*,-\delta^*,-\beta^*)$	Min.Max. Solution $(\gamma^*,\delta^*,\beta^*)$ & $(-\gamma^*,-\delta^*,-\beta^*)$
$(\gamma^*,\delta^*) = \begin{cases} (A_o^*u,\ K_o^*z) & \text{w.p. } \frac{1}{2h^*} \\ \vdots & \\ (A_{h^*-1}^*u,\ K_{h^*-1}^*z) & \text{w.p. } \frac{1}{2h^*} \\ (-A_o^*u,\ -K_o^*z) & \text{w.p. } \frac{1}{2h^*} \\ \vdots & \\ (-A_{h^*-1}^*u,\ -K_{h^*-1}^*z) & \text{w.p. } \frac{1}{2h^*} \end{cases}$ where A_i^*s, K_i^*s defined in (3.11) and h^* in (3.7) $\beta^*(u) = \eta$ where $\eta \sim N(0,\Lambda^*)$ and Λ^* in (3.12) $J^* = \dfrac{\left(\sum_{i=1}^{\ell}\sqrt{\lambda_i w_i}\right)^2}{P + \sum_{i=1}^{\ell} w_i} + \sum_{i=\ell+1}^{n} \lambda_i$ where ℓ is defined in (3.10) and w_is in (3.5)	$(\gamma^*,\delta^*) = (A_o^*u,\ K_o^*z)$ w.p. 1 $\beta^*(u) = \eta$ $\underline{J}^* = \dfrac{\left(\sum_{i=1}^{\ell}\sqrt{\lambda_i w_i}\right)^2}{P + \sum_{i=1}^{\ell} w_i} + \sum_{i=\ell+1}^{n} \lambda_i$ (notation here is the same as in the vector case (i))	R_5: $\gamma^*(u) =$ arbitrary $\beta^*(u) = {}^*(-u)$ $\delta^*(z) = 0$ $\bar{J}^* = \sum_{i=1}^{n} \lambda_i$ R_6: $\gamma^*(u) = \dfrac{1}{1-\sqrt{\frac{Q}{P}}}\tilde{A}^*u$ $\beta^*(u) = \dfrac{-\sqrt{\frac{Q}{P}}}{1-\sqrt{\frac{Q}{P}}}\tilde{A}^*u$ $\delta^*(z) = \tilde{K}^*z$ $\bar{J}^* = \dfrac{\left(\sum_{i=1}^{\tilde{\ell}}\sqrt{\lambda_i r_i}\right)^2}{(\sqrt{P}-\sqrt{Q})^2 + \sum_{i=1}^{\tilde{\ell}} r_i} + \sum_{i=\tilde{\ell}+1}^{n} \lambda_i$ where $\tilde{A}^*$, $\tilde{K}^*$, $\tilde{\ell}$ are defined in (3.48) and (3.46), respectively.

CHAPTER 4

CONCLUSION

4.1. Summary of Results

This thesis' principal objective has been to obtain saddle-point or worst-case solutions for a class of communication systems. The precise mathematical formulation of the problems has been provided in Chapter 1. Repeating briefly here, the problems involved the transmission of a sequence of independent, identically distributed Gaussian random variables over the time-discrete additive white Gaussian channel which was subjected to statistically unknown jamming noise generated by an intelligent jammer. The jammer was intelligent because he had access to the input variables. The scalar case (i) (single input-single channel) referred to the communication system in which a noiseless side channel was present for the transmission of the information about the current structure of the encoder to the decoder, in addition to the main communication channel. The vector case (i), of course, was the vector extension of the scalar case (i), in which vector input variables were transmitted over multiple channels and at the same time the vector problem still maintained the structure of the scalar problem. Besides the absence of the side channel, the scalar case (ii) and vector case (ii) were similar to the scalar case (i) and vector case (i), respectively.

In Chapter 2, we first reviewed the recent literature [3] closely related to our present work. Following that, a saddle-point solution was obtained for the scalar case (i). It appeared that the encoding policy was a mixing policy of two optimal linear decision rules on u with equal probability; in turn, the jamming policy was forced to be uncorrelated

with input signal (actually, being an independent Gaussian noise). A max.min. solution was also derived for scalar case (ii), in which the encoding policy was always a linear function of u and the jamming policy turned out to be an independent Gaussian noise. Furthermore, its max.min. value is the same as the saddle-point value in scalar case (i). We also obtained, in the same chapter, a min.max. solution for scalar case (ii) The solutions were structurally different in the three parametric regions. Nevertheless, the optimal encoding policy was always a linear function of u. Depending on the parametric regions, the general form of the jamming policy consisted of a linear function of u and an independent Gaussian random variable. In scalar case (ii), the min.max. value was always strictly greater than the max.min. value. In all cases, the optimal decoding policy was always a Bayes' estimate. Finally, Chapter 2 also provided a summary of all the scalar results, in the last section.

In Chapter 3, we tackled the vector cases in a way similar to Chapter 2. The encoding policy was restricted to the mixing policies of linear decision rules for the sake of mathematical tractability. Again, a saddle-point solution existed for vector case (i), in which the encoding policy was a randomized policy with uniform discrete probability over a set of optimal linear decision rules, and the jammer adopted an independent Gaussian random vector, which equalized the eigenvalues of channel noise. The max.min. solution of vector case (ii) had the same structure as vector case (i), besides the fact that its encoding and decoding policies were ones of possible realizations of the mixing policy of vector case (i). Then a min.max. solution was derived in an explicit closed form for parametric regions R_5

and R_6. In R_6, both the encoding and jamming policies were nonrandomized linear decision rules on u. Similar to the scalar cases, the optimum decoding policy was a Bayes' estimate in all cases. Also, some numerical examples were worked out to illustrate the spirit of the theorems of Chapter 3.

4.2. Continuation and Extension

In this section we discuss some future work which may stem from this thesis. Our continuing goal is to pursue a complete set of min.max. solutions for the vector case (ii). Outside the regions R_5 and R_6, the problem does not seem to admit an explicit closed-form solution, so that our preference would be to derive a recursive, numerical procedure for obtaining the solutions. However, we feel that the min.max. value of J, $\bar{J}^*$ is always strictly greater than the max.min. value of J, $\underline{J}^*$; that is, the saddle-point solution never exists.

The scalar case (ii) treated in this thesis can be extended to a more complicated version analogous to the communication system as depicted in Figure 4.1. The only difference between the system in [8] and the extended system in Figure 4.1 is that the jammer in [8] taps the output of the encoder, $\{x_{t_j}\}$, instead of tapping the input, $\{m_t\}$. The extended system here follows the same structure as the communication system in [8] except for this difference. So, the extended problem is to consider the transmission of a sequence of random variables generated by a Gaussian discrete-time Markov process, $\{m_t\}$, over a time-discrete additive white Gaussian channel in the presence of an intelligent jammer and an independent white Gaussian variable

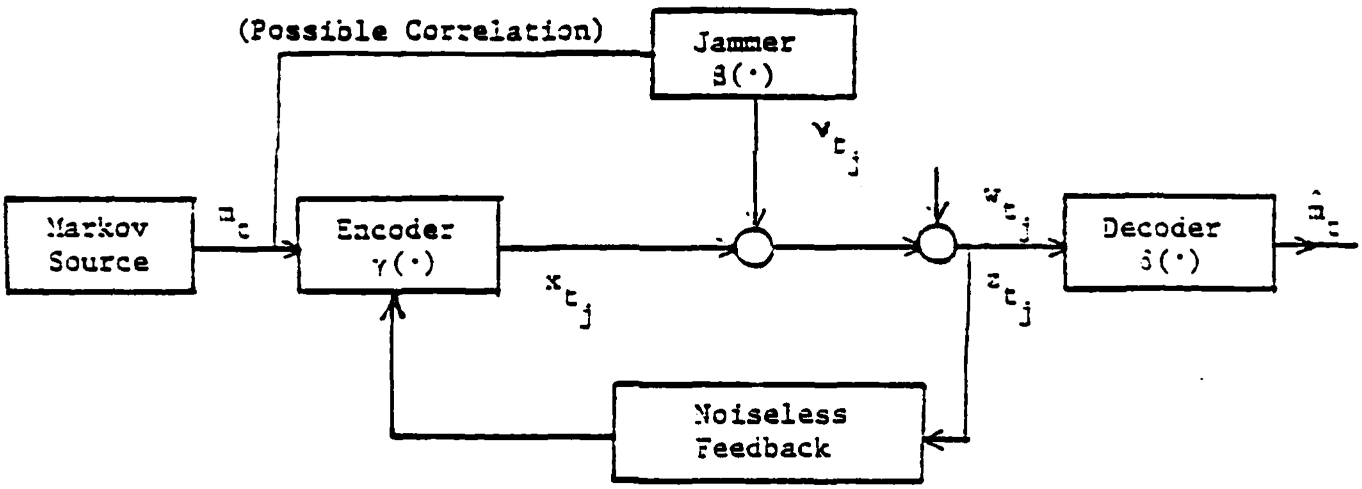

Figure 4.1. A Schematic description of the extended communication system.

$\{w_{t_j}\}$, i.e., $\{m_t\}$ is described by

$$m_{t+1} = A_t m_t + B_t \xi_t \quad , \quad t=0,1,\ldots$$

where $\{t_j : j=1..n\}$ are the sub-time indices between t and t+1, $\{\xi_t\}$ is a q-dimensional standard Gaussian white noise process, m_o is Gaussian with mean zero and variance p_o, and B_t is a 1xq vector. During the interval between the arrival of successive input variables, m_t and m_{t+1}, the channel can be used n times such that each time the encoding policy will satisfy the channel input power constraint, i.e., $E(x^2_{t_j}) \leq P$. The encoder can also access the channel output $\{z_{t_j}\}$ through a noiseless feedback link. Overall, the new problem is still to seek the saddle-point solution, or if it does not exist, the worst case solutions by both min.max. and max.min. approaches.

In [8], two separate problems have been solved under the following specifications: (a) the jamming noise $\{v_{t_j}\}$ is correlated with the encoded signal $\{x_{t_j}\}$, and (b) the jamming noise $\{v_{t_j}\}$ is taken to be independent of the encoded signals $\{x_{t_j}\}$ and all other random processes. Saddle-point solutions exist for both problems. The general form of encoding policies is a linear innovation process on the input and the channel output $\{z_{t_j}\}$, and the decoders are always Bayes' estimates for the corresponding situation. However, the jamming policies admit different structures for these two problems. In problem (a), the jamming policy consists of a linear function of the encoded signal $\{x_{t_j}\}$ and an independent, white Gaussian process. For (b), the jamming policy consists of only an independent white Gaussian process. By recalling the similarity between scalar case (ii) and scalar case (iii), (Table 2.1) and the Gaussian property of the problem, we feel that the min.max. and max.min. approaches in the extended problem should lead to the same solutions

as the saddle-point solutions of (a) and (b) in [8], respectively. Hence, the saddle-point solution is not expected to exist in the extended problem depicted in Figure 4.1.

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